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Linking the mixing times of random walks on static and dynamic random graphs

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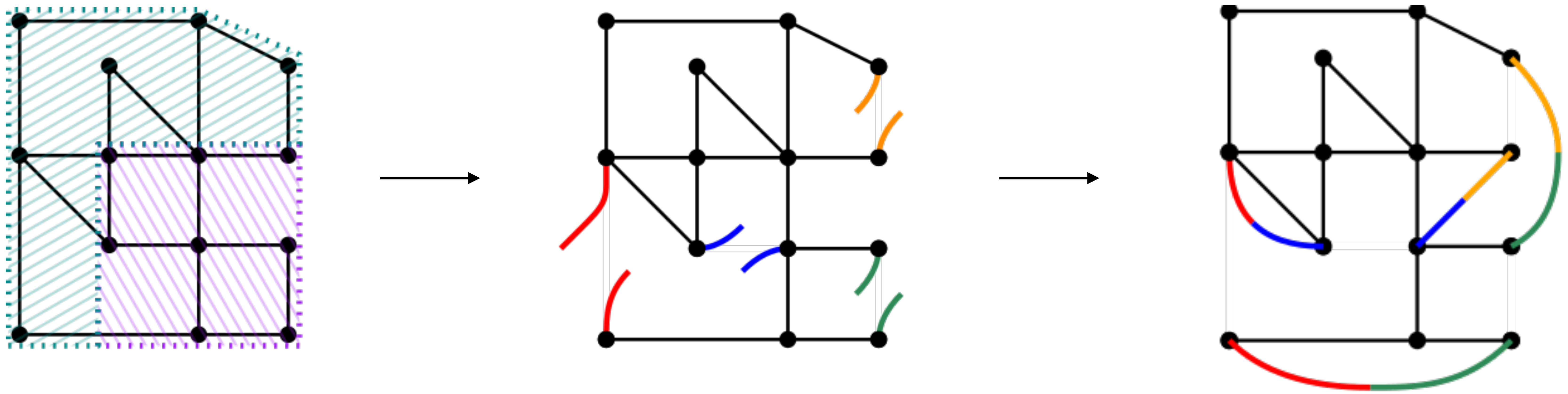
Setting

- **Non-backtracking** random walk on a dynamical graph.
- Graph initially drawn according to the **configuration model.**
- Graph evolution described by **rewiring of edges.**



- Degree sequence and graph evolution subject to mild **regularity conditions.**
  - **Degree sequence:** sparse, tree-like, degrees ≥ 2.
  - **Graph evolution:** “dynamical self-avoidance”, lack of bias for rewiring.

Link between the dynamic and static mixing time

- Suppose that regularity conditions hold and $t = O(\log n)$. Then the following holds with high probability in x and ξ:

$$\mathscr{D}^{\mathrm{dyn}}_{x,\xi}(t) = \mathbb{P}(\tau > t)\ \mathscr{D}^{\mathrm{stat}}_{x,\xi}(t) + o_{\mathbb{P}}(1)\,.$$

- Random variable τ is the first time the random walk steps over a rewired edge.

Proof idea

Coupling between the dynamic and static situation

- As long as the random walk does not step over an edge that has changed compared to the initial graph,

the random walk on the dynamic graph

before stepping over a rewired edge

<u>is essentially the same as</u>

the random walk on the initial graph.

- When the **dynamic random walk steps over a rewired edge** the walk on the **static initial graph makes a random jump**.

- **Situations where this approximation fails in logarithmic time (or faster) are rare**, due to regularity conditions.

Concrete examples

General approach

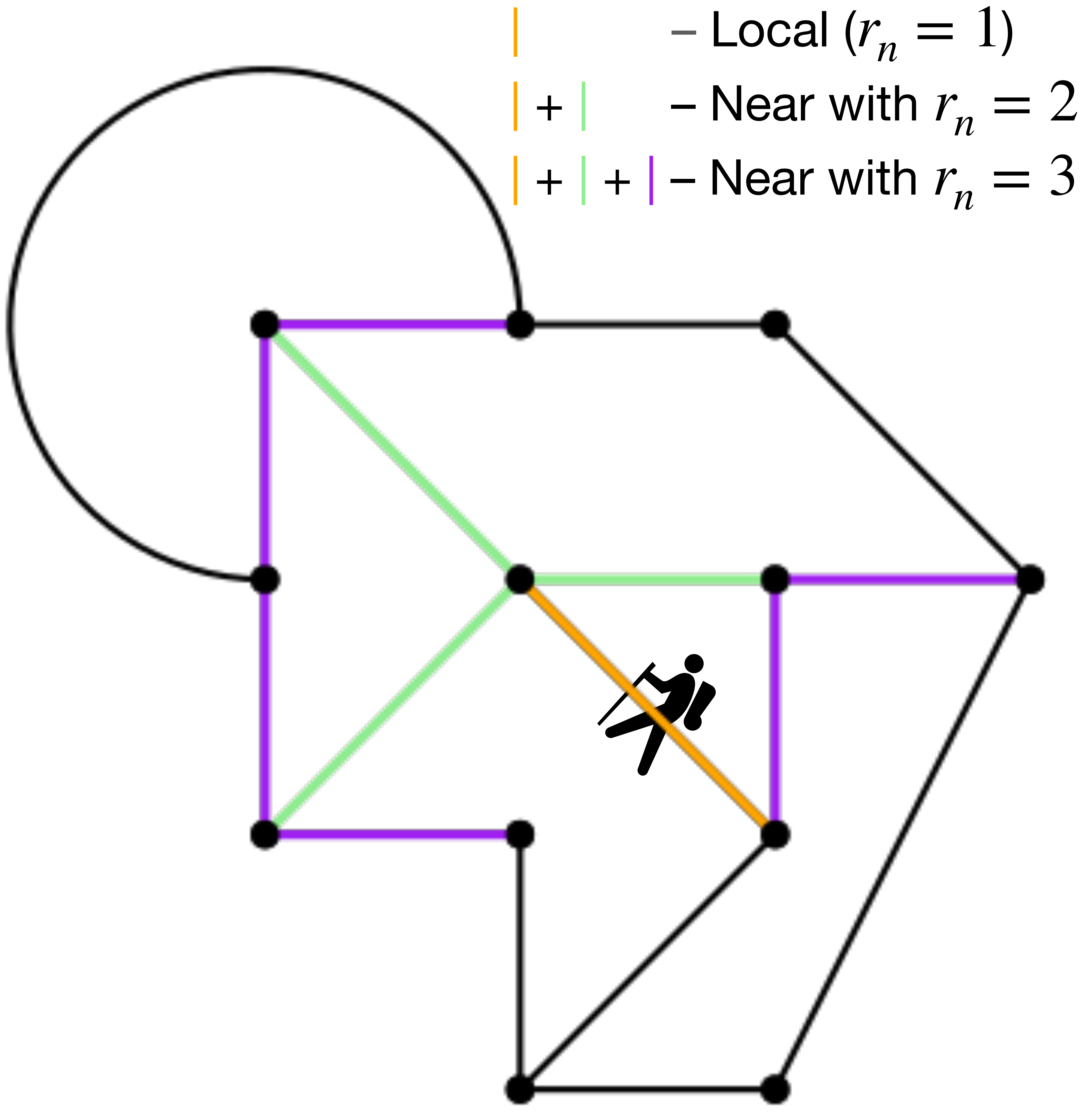
- Recall: $$\mathscr{D}^{\text{dyn}}_{x,\xi}(t) = \mathbb{P}(\tau > t)\ \mathscr{D}^{\text{stat}}_{x,\xi}(t) + o_{\mathbb{P}}(1)$$
- $\mathscr{D}^{\text{stat}}_{x,\xi}(t)$-term:
  - We use the results of Ben-Hamou and Salez [1]. This introduces more strict conditions on the degrees than required by the coupling argument.
- $\mathbb{P}(\tau > t)$-term: precise combinatorial estimates.
  - Idea: given the rewiring mechanism, count the opportunities when the random walk can step over a rewired edge.

[1] Anna Ben-Hamou. Justin Salez. "Cutoff for nonbacktracking random walks on sparse random graphs." Ann. Probab. 45(3) 1752-1770, May 2017.

Concrete examples

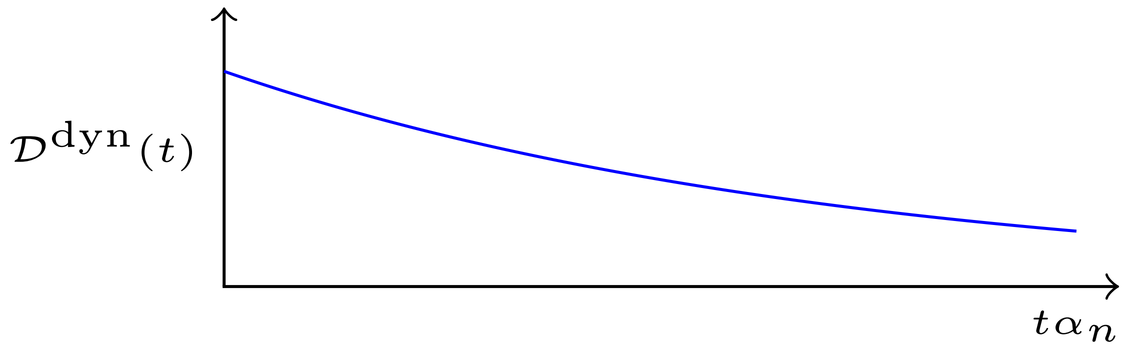
Location-dependents sets of edges

- Near-set around half-edge X
  - Set of edges that the NBRW can traverse in r_n steps starting from X, given that the graph remains unchanged.
  - If $r_n = 1$, we call it a **local-set.**
  - If $r_n = \mathrm{diam}(G)$, we call it the **global-set**

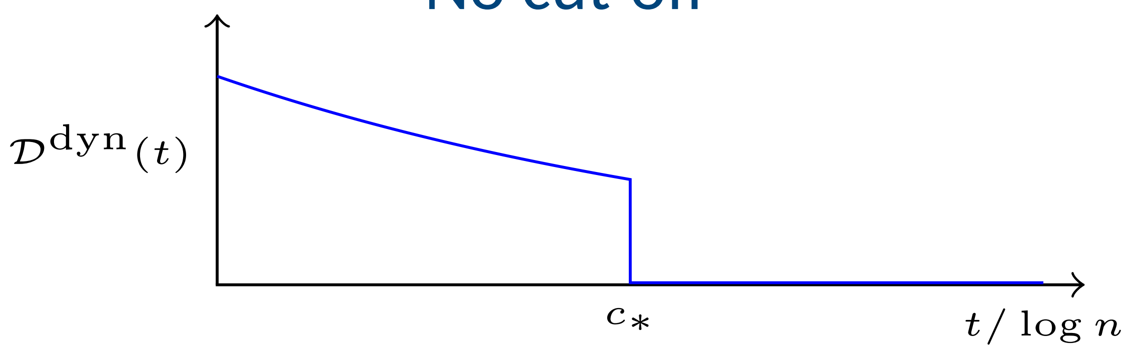


Concrete examples

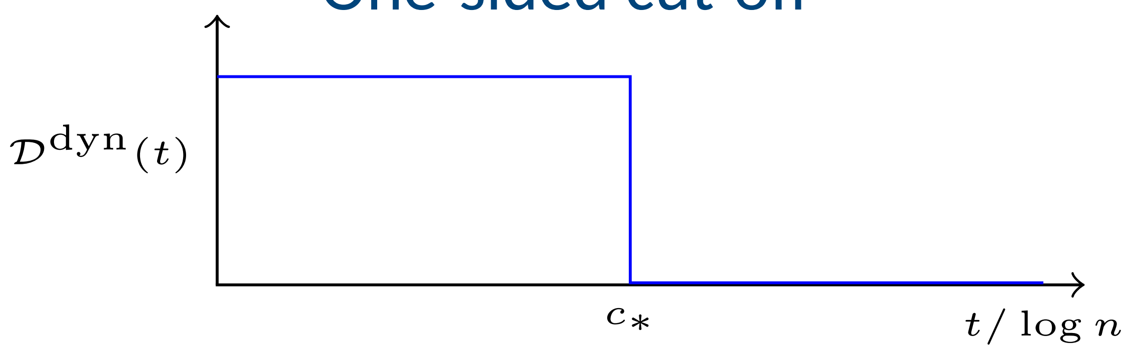
Overview of results



No cut-off

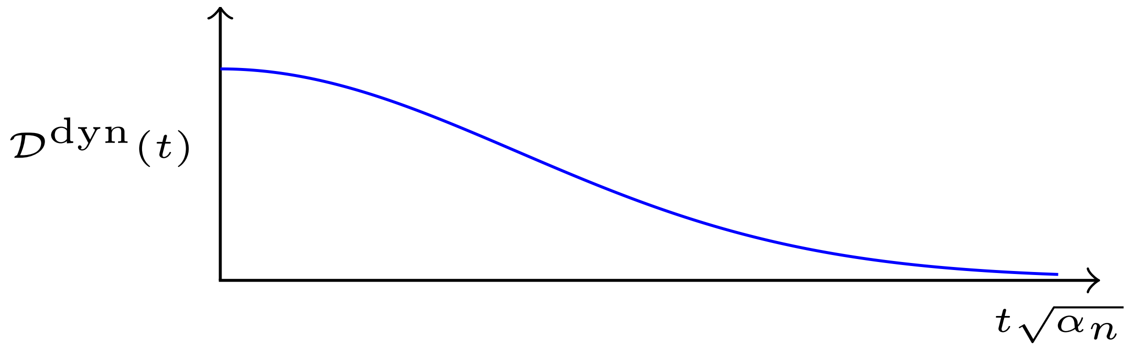


One-sided cut-off



Two-sided cut-off

Trichotomy of $\mathcal{D}^{\mathrm{dyn}}_{x,\xi}(t)$ for **local-to-global** rewiring.



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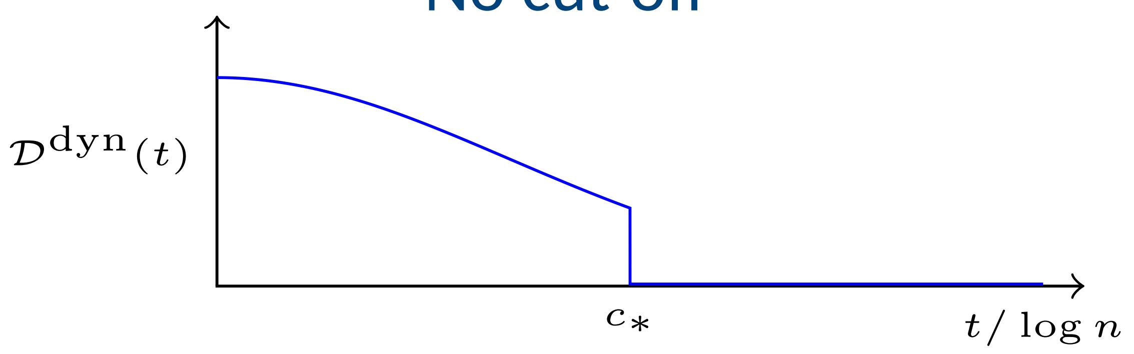
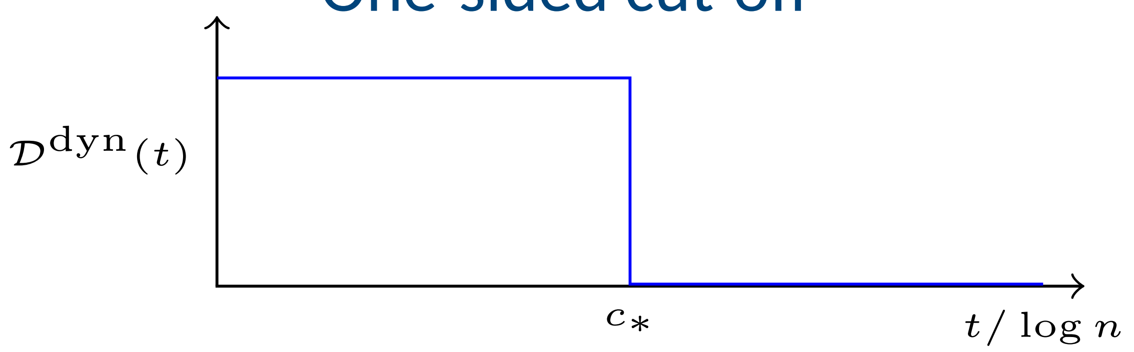
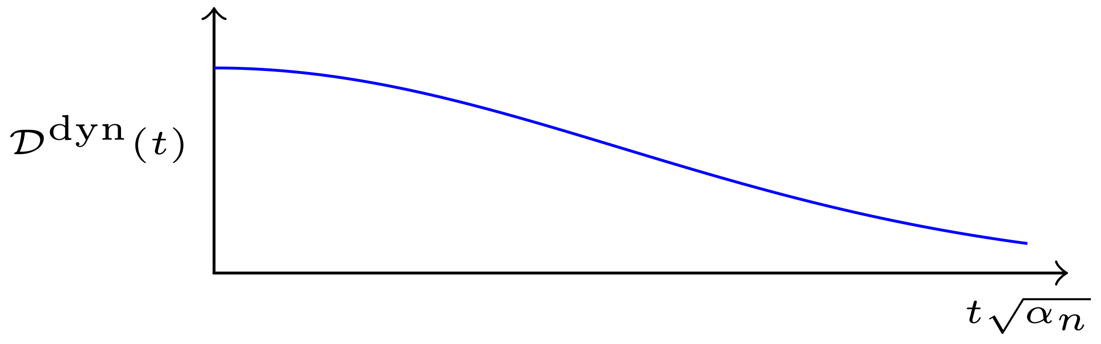


One-sided cut-off

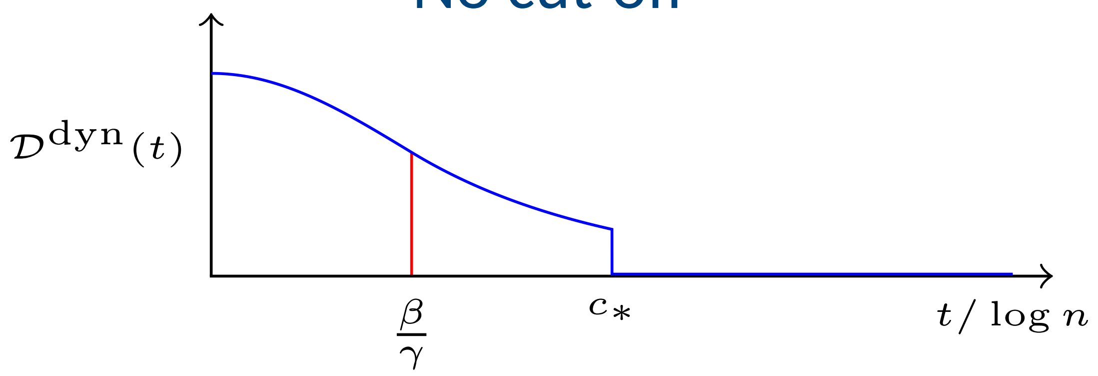


Two-sided cut-off

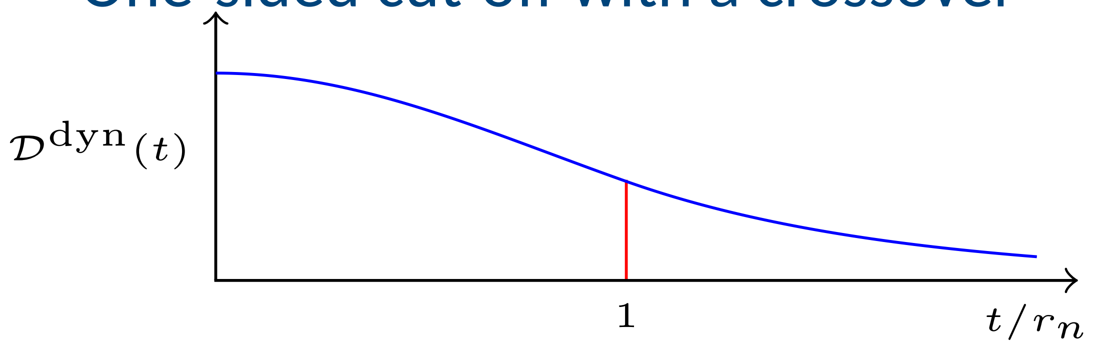
Trichotomy of $\mathcal{D}^{\mathrm{dyn}}_{x,\xi}(t)$ for **global-to-global** rewiring.



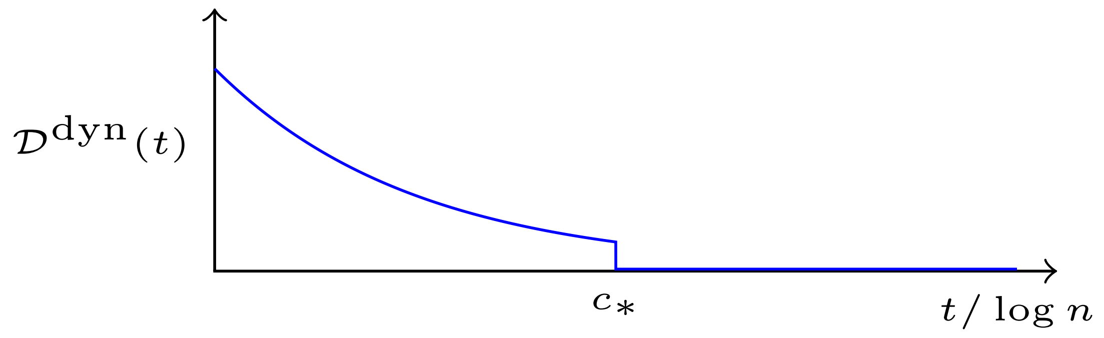
No cut-off



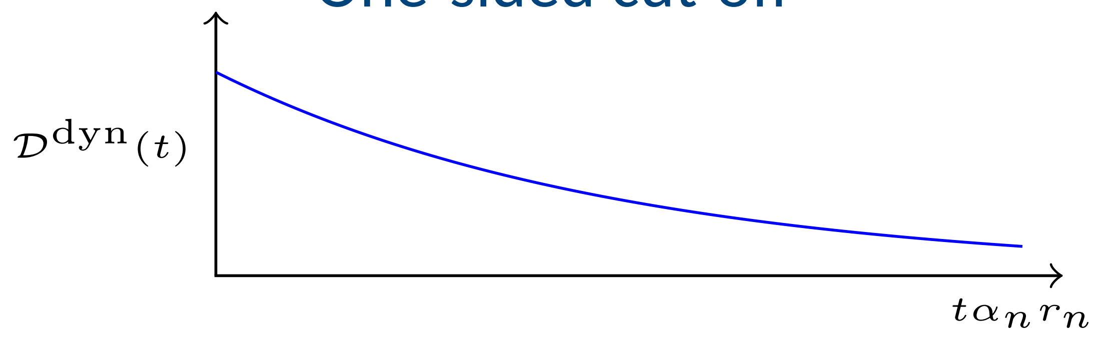
One-sided cut-off with a crossover



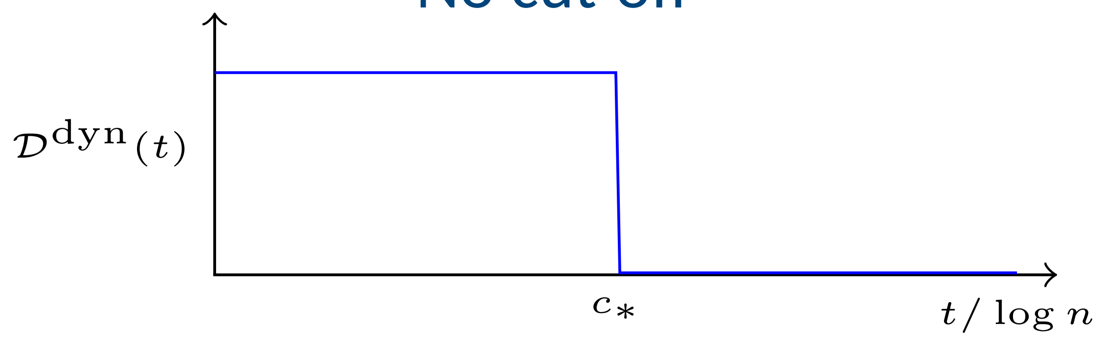
Crossover, but without a cut-off



One-sided cut-off



No cut-off



Two-sided cut-off

Hexachotomy of $\mathcal{D}^{\mathrm{dyn}}_{x,\xi}(t)$ for **near-to-global** rewiring.
The red line marks a change in the shape of the curve.

Concrete examples

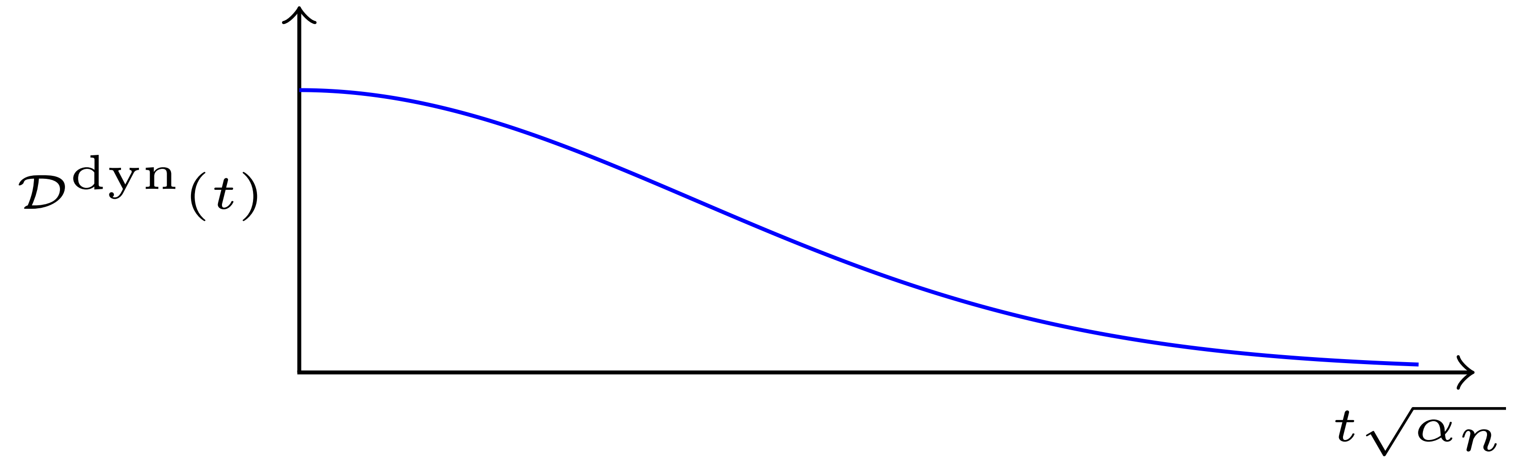
Global-to-global

- Rewiring happens “everywhere”.

One parameter: α_n

(1) *If* $\lim_{n\to\infty} \alpha_n (\log n)^2 = \infty$, *then*

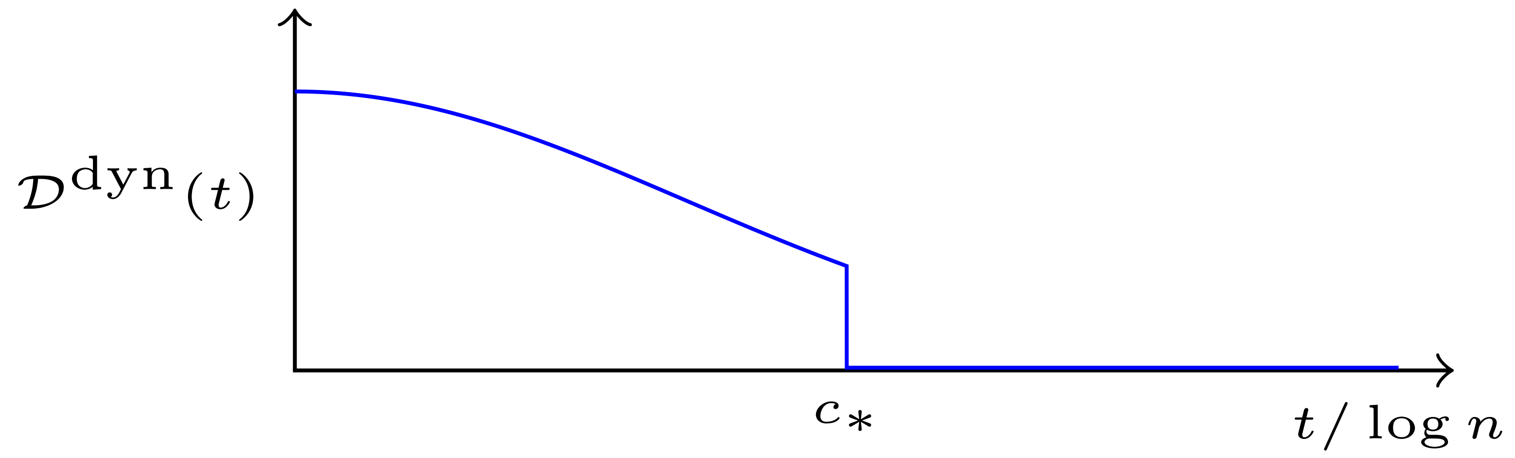
$$\mathcal{D}^{\mathrm{dyn}}_{x,\xi}\big(\lfloor c/\sqrt{\alpha_n}\rfloor\big) = o_{\mathbb{P}}(1) + \mathrm{e}^{-c^2/2}, \quad c \in [0,\infty).$$



Regime (1): (1.24).

(2) *If* $\lim_{n\to\infty} \alpha_n (\log n)^2 = \gamma \in (0,\infty)$, *then*

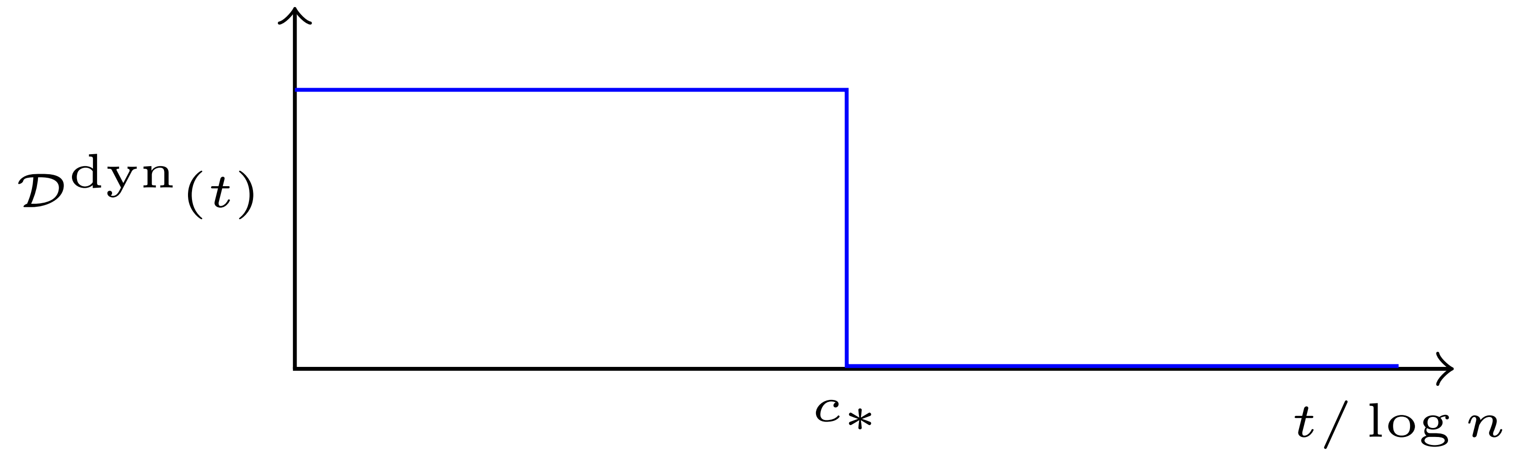
$$\mathcal{D}^{\mathrm{dyn}}_{x,\xi}\big(\lfloor c\log n\rfloor\big) = o_{\mathbb{P}}(1) + \begin{cases} \mathrm{e}^{-\gamma c^2/2}, & c \in [0,c_*), \\ 0, & c \in (c_*,\infty). \end{cases}$$



Regime (2): (1.25).

(3) *If* $\lim_{n\to\infty} \alpha_n (\log n)^2 = 0$, *then*

$$\mathcal{D}^{\mathrm{dyn}}_{x,\xi}\big(\lfloor c\log n\rfloor\big) = o_{\mathbb{P}}(1) + \begin{cases} 1, & c \in [0,c_*), \\ 0, & c \in (c_*,\infty). \end{cases}$$



Regime (3): (1.26).

Concrete examples

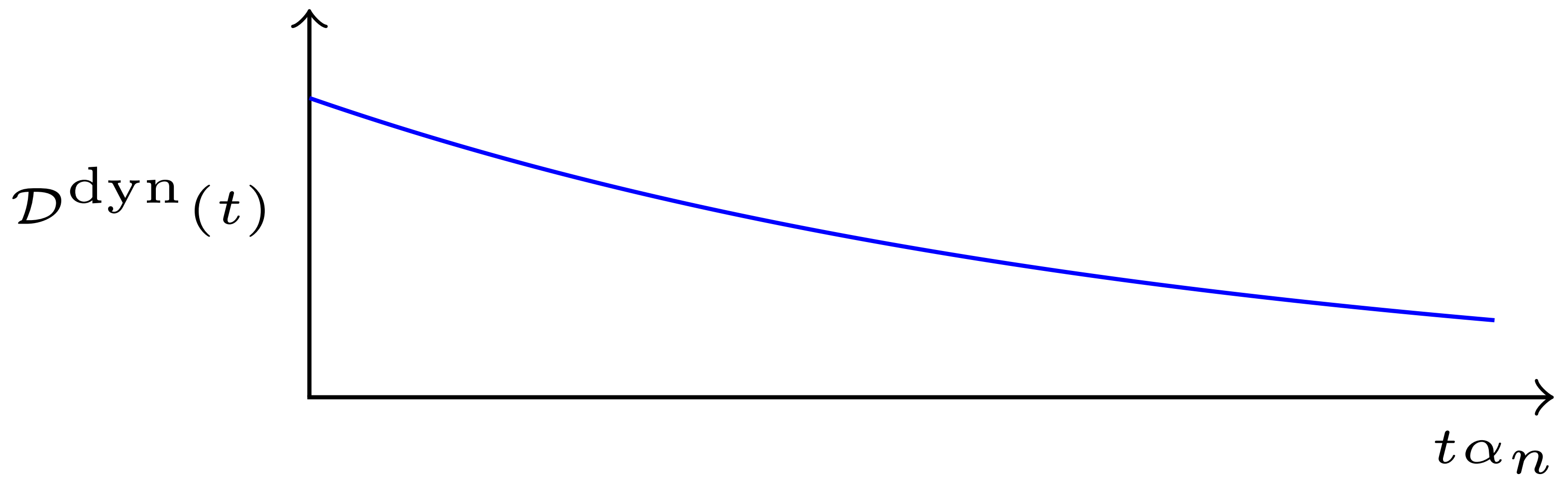
Local-to-global

- Rewiring happens “right under my feet”.

One parameter: α_n

(1) *If* $\lim_{n\to\infty} \alpha_n \log n = \infty,$ *then*

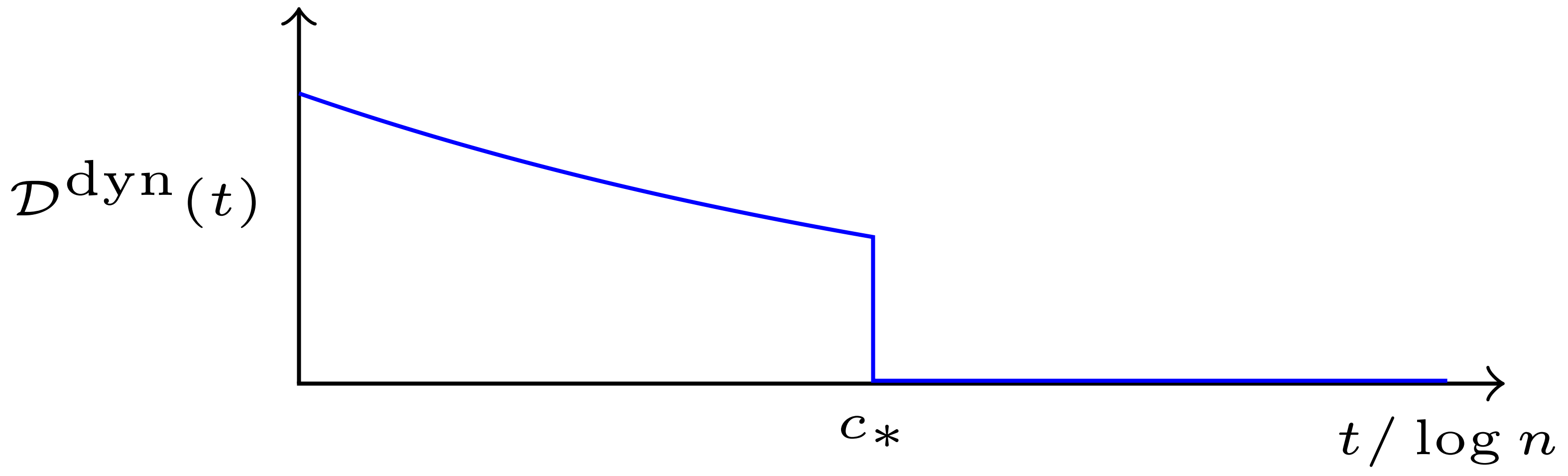
$$\mathcal{D}^{\mathrm{dyn}}_{x,\xi}\big(\lfloor c/\alpha_n \rfloor\big) = o_{\mathbb{P}}(1) + \mathrm{e}^{-c}, \quad c \in [0,\infty).$$



Regime (1): (1.15).

(2) *If* $\lim_{n\to\infty} \alpha_n \log n = \gamma \in (0,\infty),$ *then*

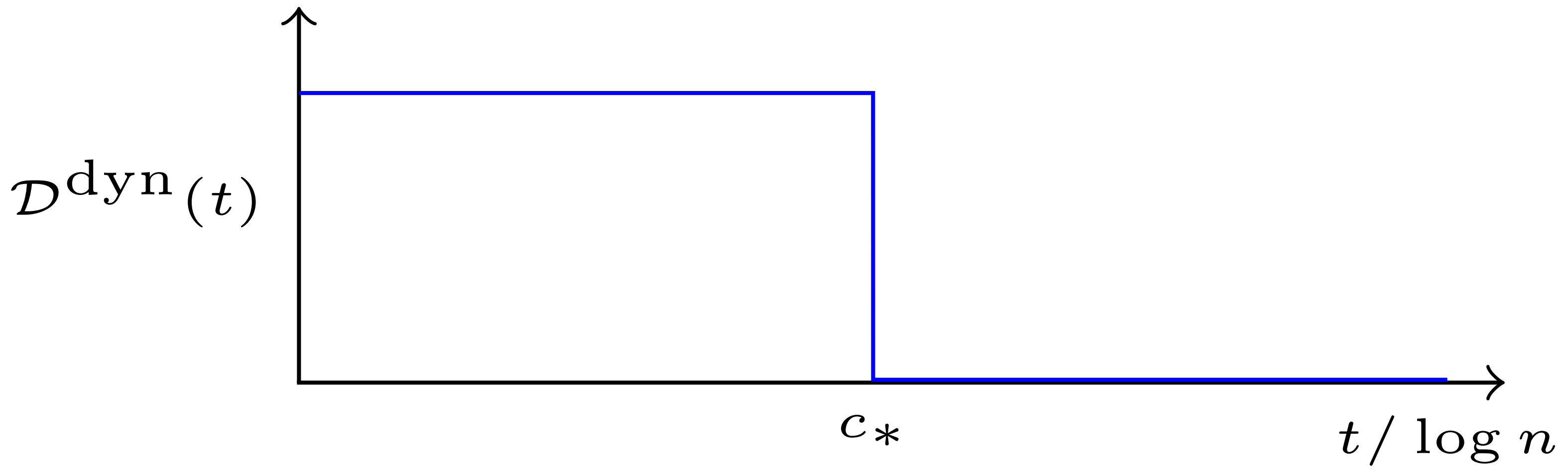
$$\mathcal{D}^{\mathrm{dyn}}_{x,\xi}\big(\lfloor c \log n \rfloor\big) = o_{\mathbb{P}}(1) + \begin{cases} \mathrm{e}^{-\gamma c}, & c \in [0, c_*), \\ 0, & c \in (c_*, \infty). \end{cases}$$



Regime (2): (1.16).

(3) *If* $\lim_{n\to\infty} \alpha_n \log n = 0,$ *then*

$$\mathcal{D}^{\mathrm{dyn}}_{x,\xi}\big(\lfloor c \log n \rfloor\big) = o_{\mathbb{P}}(1) + \begin{cases} 1, & c \in [0, c_*), \\ 0, & c \in (c_*, \infty). \end{cases}$$

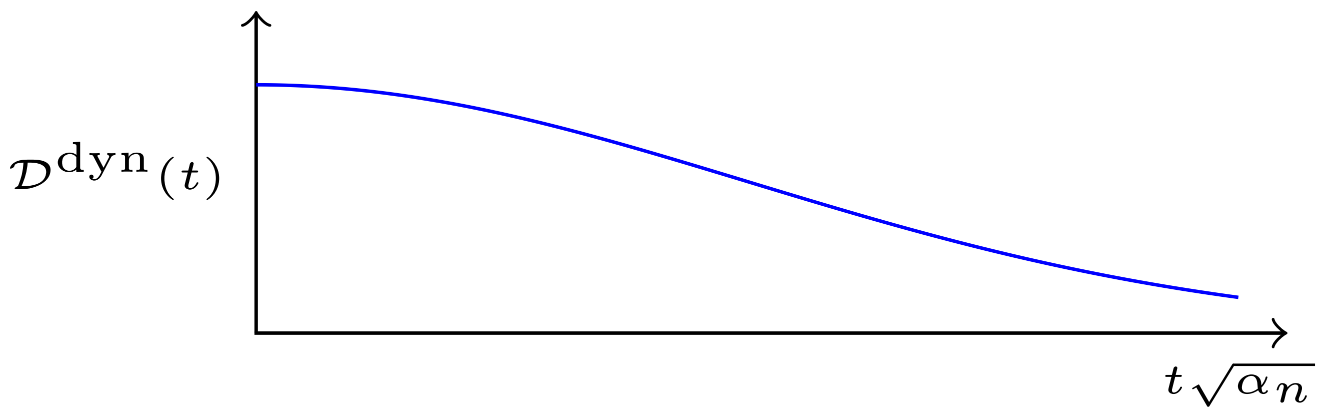


Regime (3): (1.17).

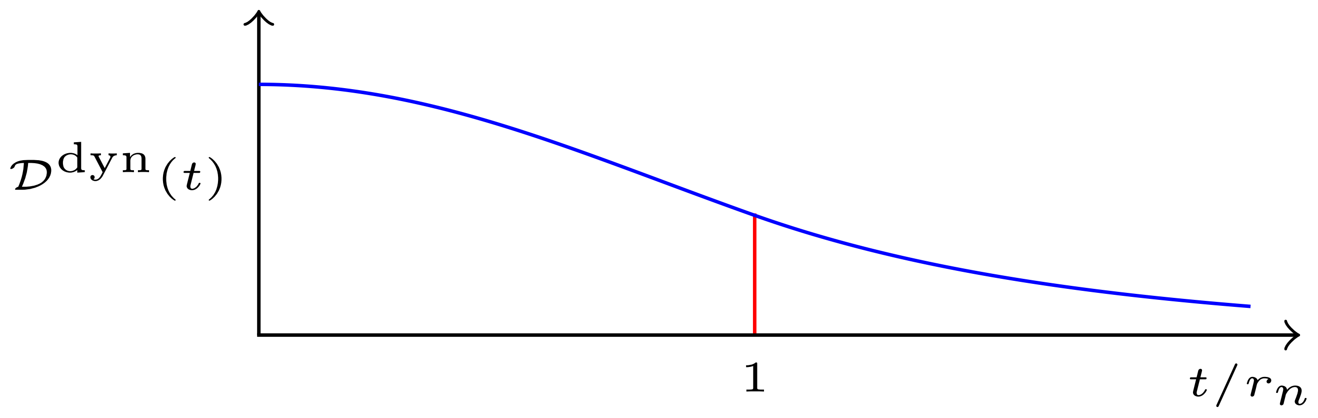
Concrete examples

Near-to-global

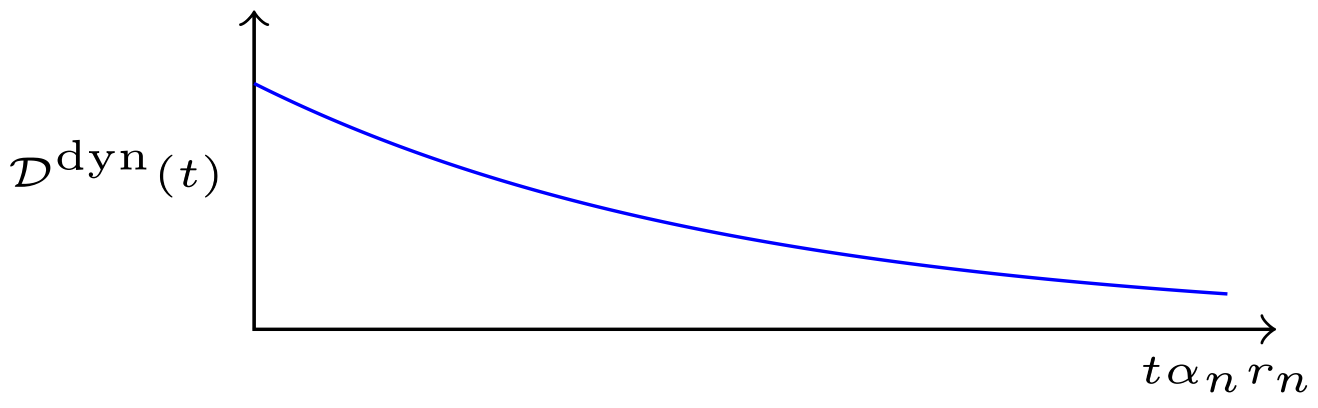
- Rewiring happens in the near-set -> **shortcuts**. Two parameters: α_n, r_n.
- Non-markovian, interpolates between the previous two.



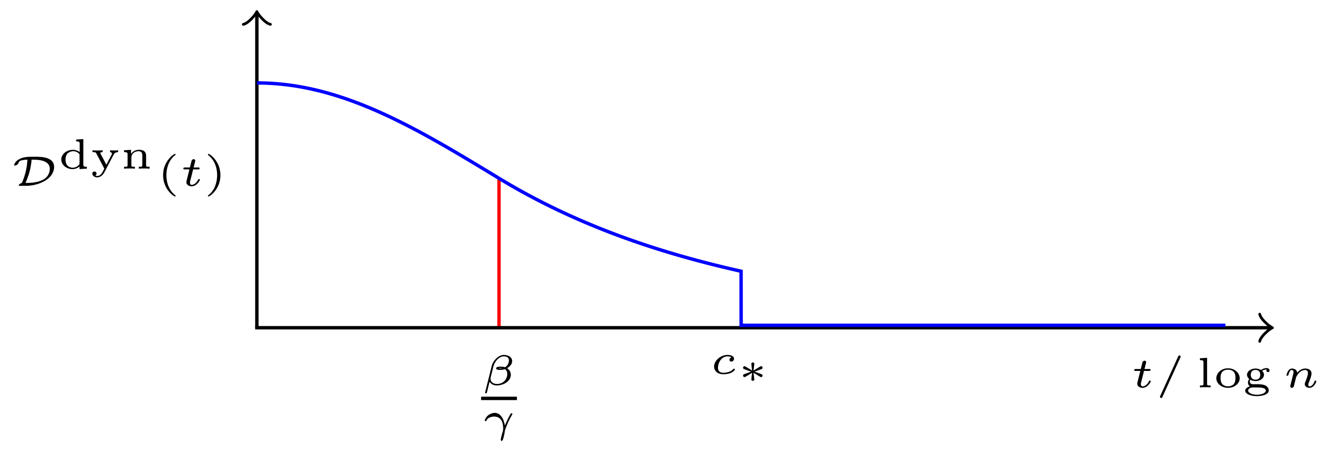
Regime 1(a): (1.18).



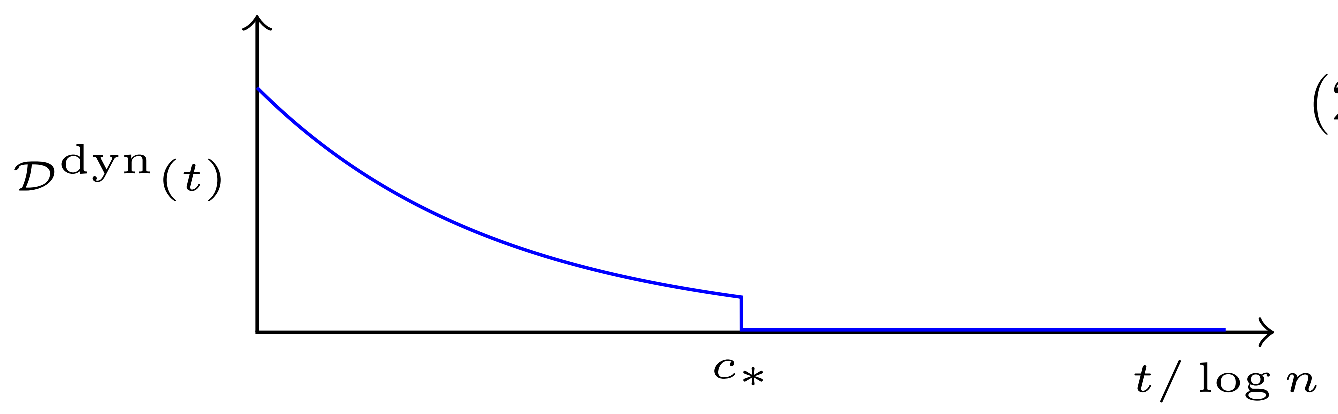
Regime 1(b): (1.19).



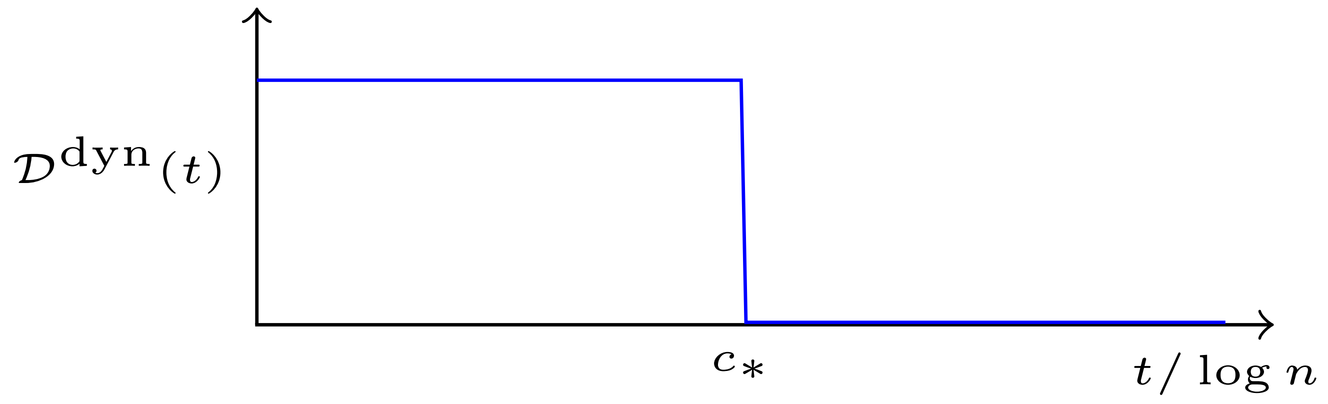
Regime 1(c): (1.20).



Regime 2(b): (1.21).



Regime 2(c): (1.22).



Regime 3(c): (1.23).

Case 2(b):

- **Cross-over** in the decay of $\mathscr{D}^{\mathrm{dyn}}(t)$

(2) *If* $\lim_{n\to\infty}\alpha_n r_n \log n = \gamma \in (0,\infty)$ *and*

(b) $\lim_{n\to\infty}\alpha_n r_n^2 = \beta \in (0,\infty)$, *then*

$$\mathcal{D}^{\mathrm{dyn}}_{x,\xi}\big(\lfloor c\log n\rfloor\big) = o_{\mathbb{P}}(1) + \begin{cases} \mathrm{e}^{-(\gamma c)^2/2\beta}, & c\in[0,\beta/\gamma],\\ \mathrm{e}^{-(2\gamma c-\beta)/2}, & c\in(\beta/\gamma,c_*),\\ 0, & c\in(c_*,\infty).\end{cases}$$

Take-away messages

- **General framework** that ties together mixing properties of non-backtracking random walks on dynamic and static configuration model graphs.

- **Applications of this framework to concrete models:**

  - Global-to-global: previously observed **trichotomy**, this time established under weaker assumptions.

  - Local-to-global: **trichotomy** similar to global-to-global.

  - Near-to-global: **hexachotomy** (six-way split) in mixing profiles, a kink in the mixing profile in one of the regimes.

Thank you for your attention.