Universiteit Leiden
Mathematical Institute

NETWORKS
NETWORKS is a project of
University of Amsterdam
Eindhoven University of Technology
Leiden University
Center for Mathematics and
Computer Science (CWI)

Mixing of fast random walks on dynamic random permutations

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Model

Underlying geometry: Dynamic permutation

- Take $n \in \mathbb{N}$ and define the sequence $\left(\Pi_n(t)\right)_{t=0}^{\infty}$ such that:
  - $\Pi_n(0) = \mathtt{Id} \in S_n$
  - $\forall t \geq 1 : \Pi_n(t) = \Pi_n(t-1) \circ (a, b)$,

    where (a, b) is transposition chosen according to a given rule

- Dynamic rules under consideration:
  - Transpositions of elements on different cycles picked u.a.r. (coagulation-only)
  - Transpositions chosen u.a.r. (coagulation-fragmentation)

Model

Stochastic process: Infinite-speed random walk (ISRW)

Definition 1.6 [Infinite-speed random walk on Π_n] Fix Π_n and an element $v_0 \in [n]$. Recall that $\gamma_v(\Pi_n(t))$ is the cycle of $\Pi_n(t)$ that contains v. The infinite-speed random walk (ISRW) starting from v_0 is the random process $X_n^{v_0} = (X_n^{v_0}(t))_{t\in\mathbb{N}_0}$ on $[n]$ with initial distribution given by

$$\mu^{X_n^{v_0}}(0) = \left(\mu_w^{X_n^{v_0}}(0)\right)_{w\in[n]}, \tag{1.4}$$

where

$$\mu_w^{X_n^{v_0}}(0) = \begin{cases} \frac{1}{|\gamma_w(\Pi_n(0))|}, & w \in \gamma_{v_0}(\Pi_n(0)), \\ 0, & w \notin \gamma_{v_0}(\Pi_n(0)), \end{cases} \tag{1.5}$$

and with distribution at time $t \in \mathbb{N}$ given by

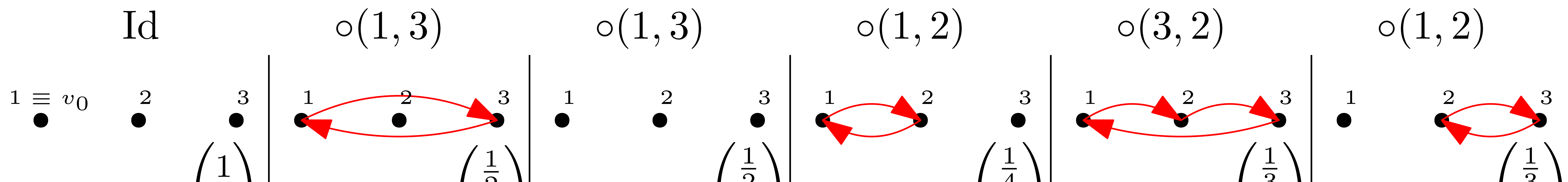
$$\mu^{X_n^{v_0}}(t) = \left(\mu_w^{X_n^{v_0}}(t)\right)_{w\in[n]}, \tag{1.6}$$

where

$$\mu_w^{X_n^{v_0}}(t) = \frac{1}{|\gamma_w(\Pi_n(t))|} \sum_{u\in\gamma_w(\Pi_n(t))} \mu_u^{X_n^{v_0}}(t-1). \tag{1.7}$$

Model

Example: ISRW on a dynamic permutation



$$\mu^{X_n^{v_0}}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mu^{X_n^{v_0}}(1) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \mu^{X_n^{v_0}}(2) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \mu^{X_n^{v_0}}(3) = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} \quad \mu^{X_n^{v_0}}(4) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad \mu^{X_n^{v_0}}(5) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

The main question

- Let v_0 denote the starting vertex of the ISRW. How does the mixing profile

$$\mathcal{D}_n^{v_0}(t) = \|\mu(t) - \mu^{\text{stat}}\|_{\text{TV}} \in [0,1]$$

  evolve in time?

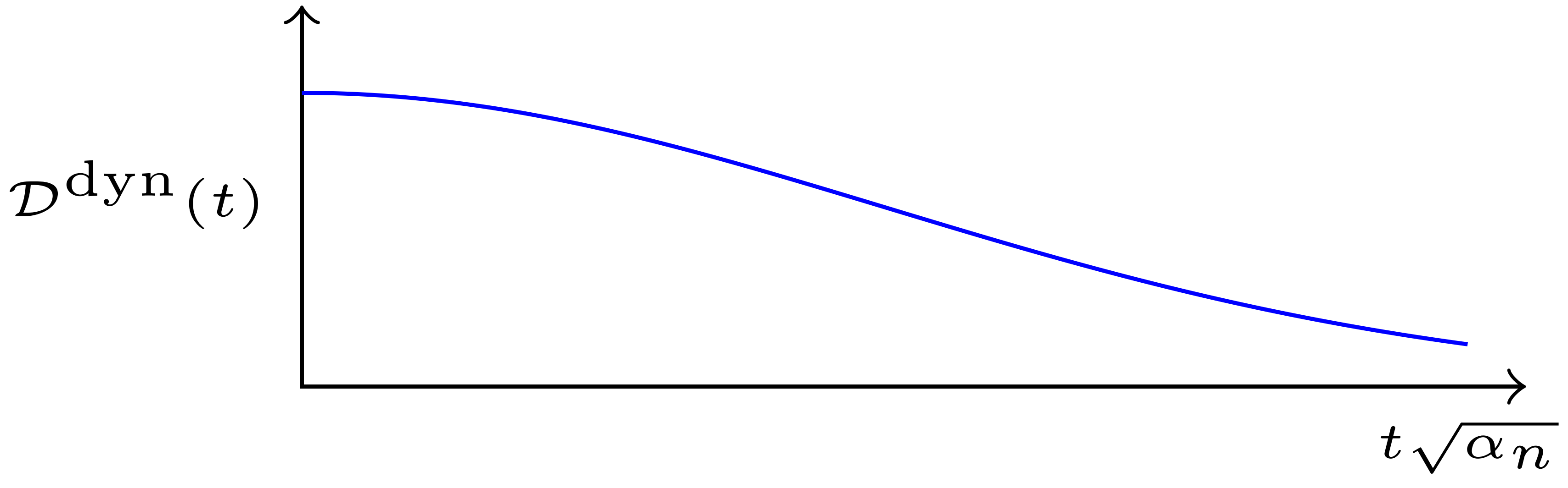
- $\|X - Y\|_{\text{TV}}$ denotes the total variation distance between prob. measures X,Y;

$$\|X - Y\|_{\text{TV}} = \sup_{A \in \Omega} |X(A) - Y(A)| \stackrel{!}{=} \frac{1}{2}\|X - Y\|_1$$ (in countable prob. spaces)

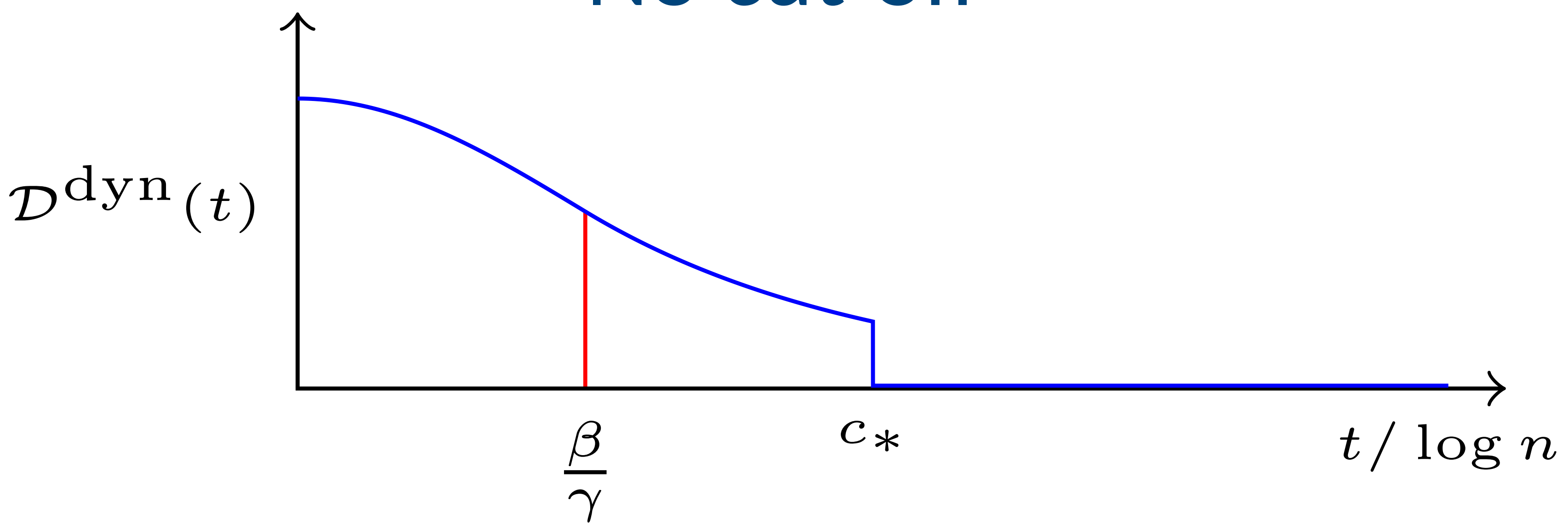
- $\mu(t)$ is the ISRW distribution at time t

- $\mu^{\text{stat}} = \text{Unif}(\{1, \ldots, n\})$ is the stationary distribution of the ISRW;

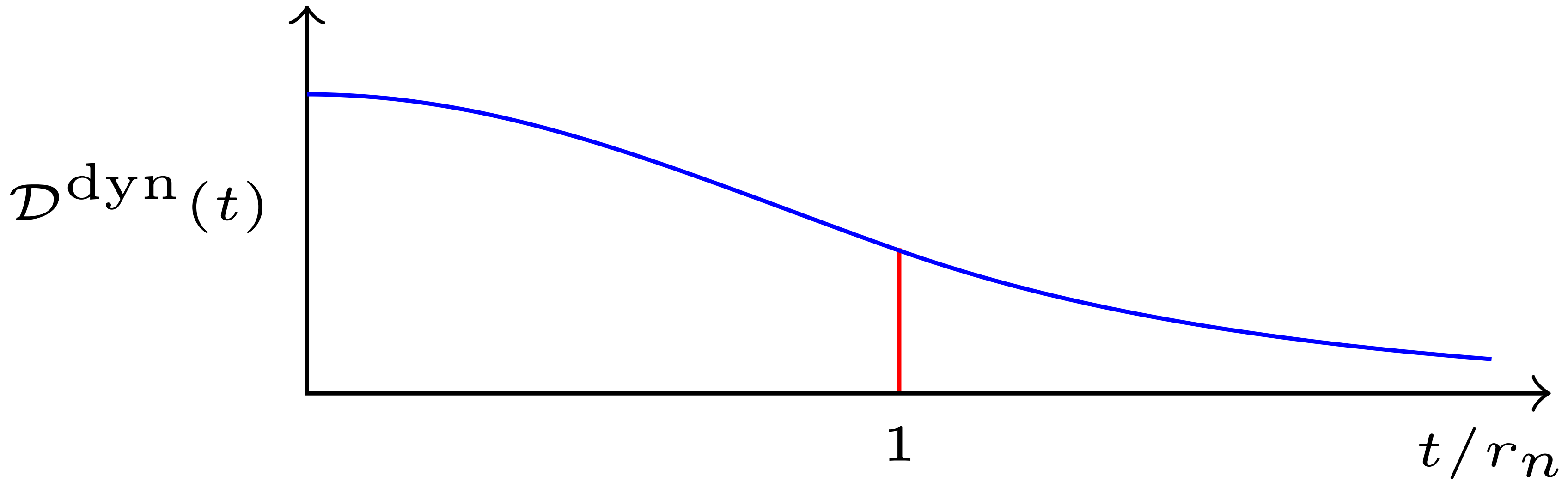
Mixing profile zoo - typical exhibits



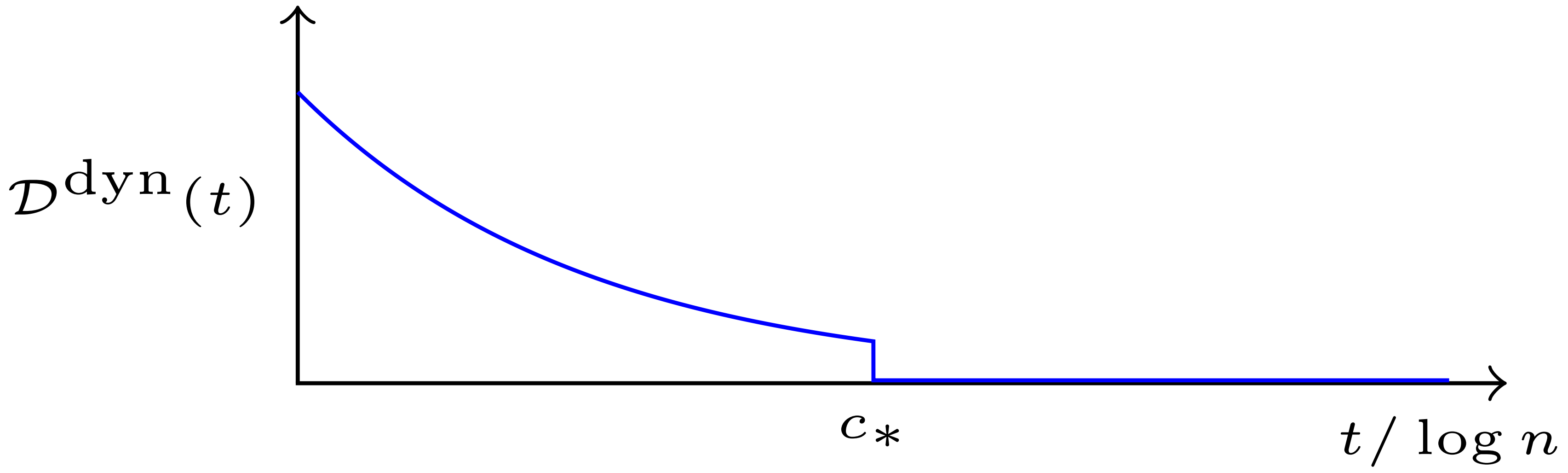
No cut-off



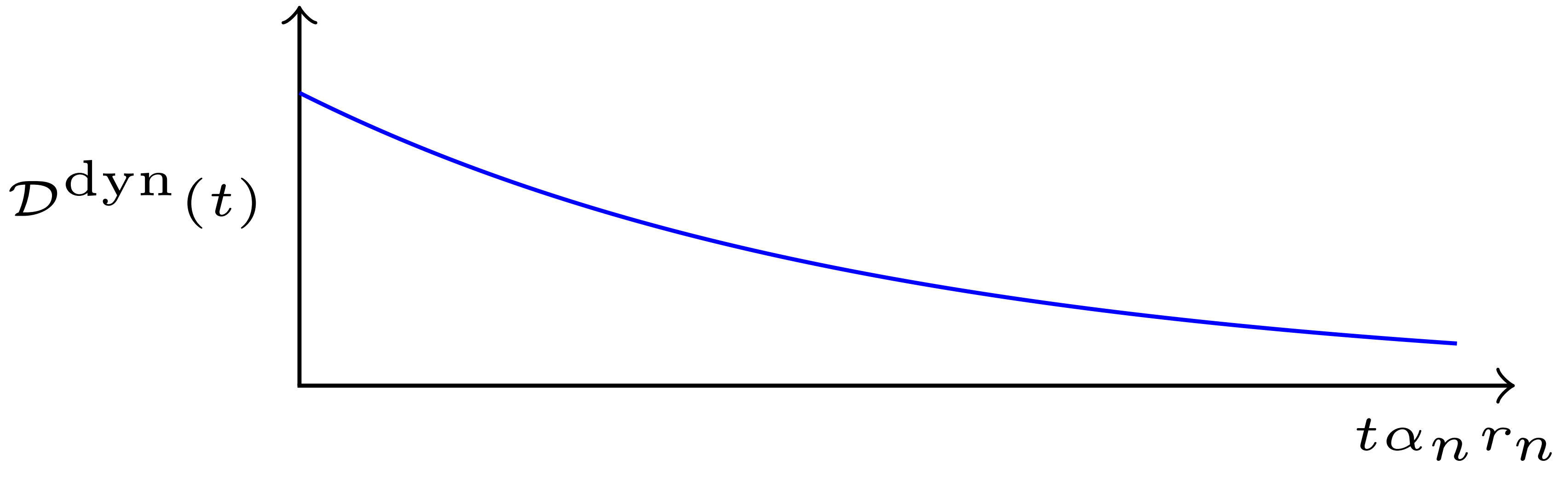
One-sided cut-off with a crossover



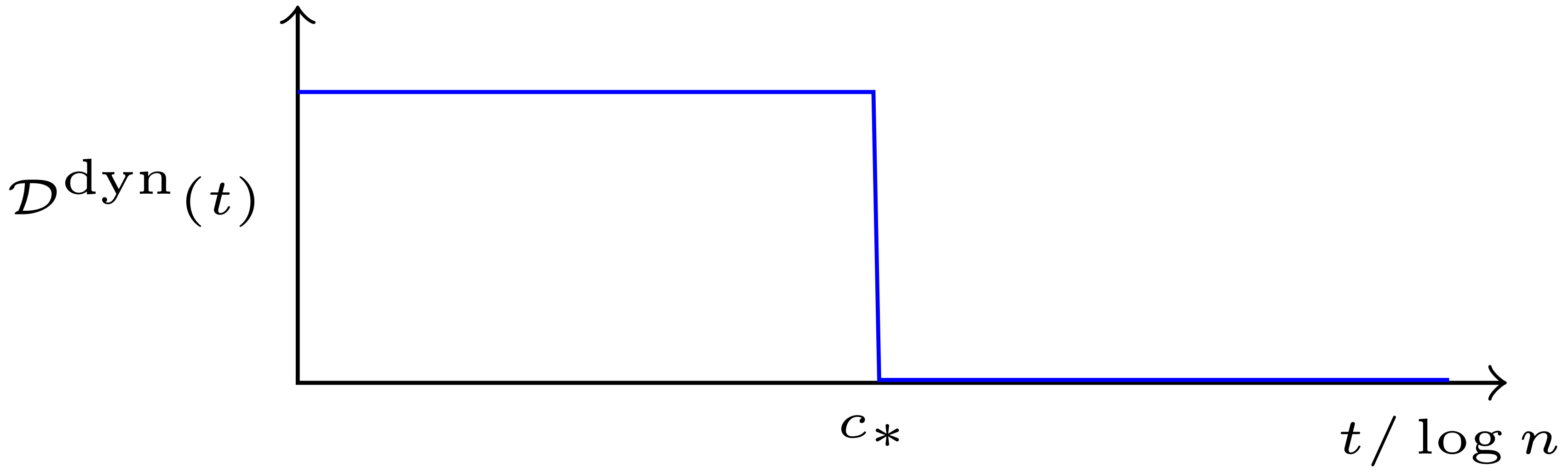
Crossover, but without a cut-off



One-sided cut-off



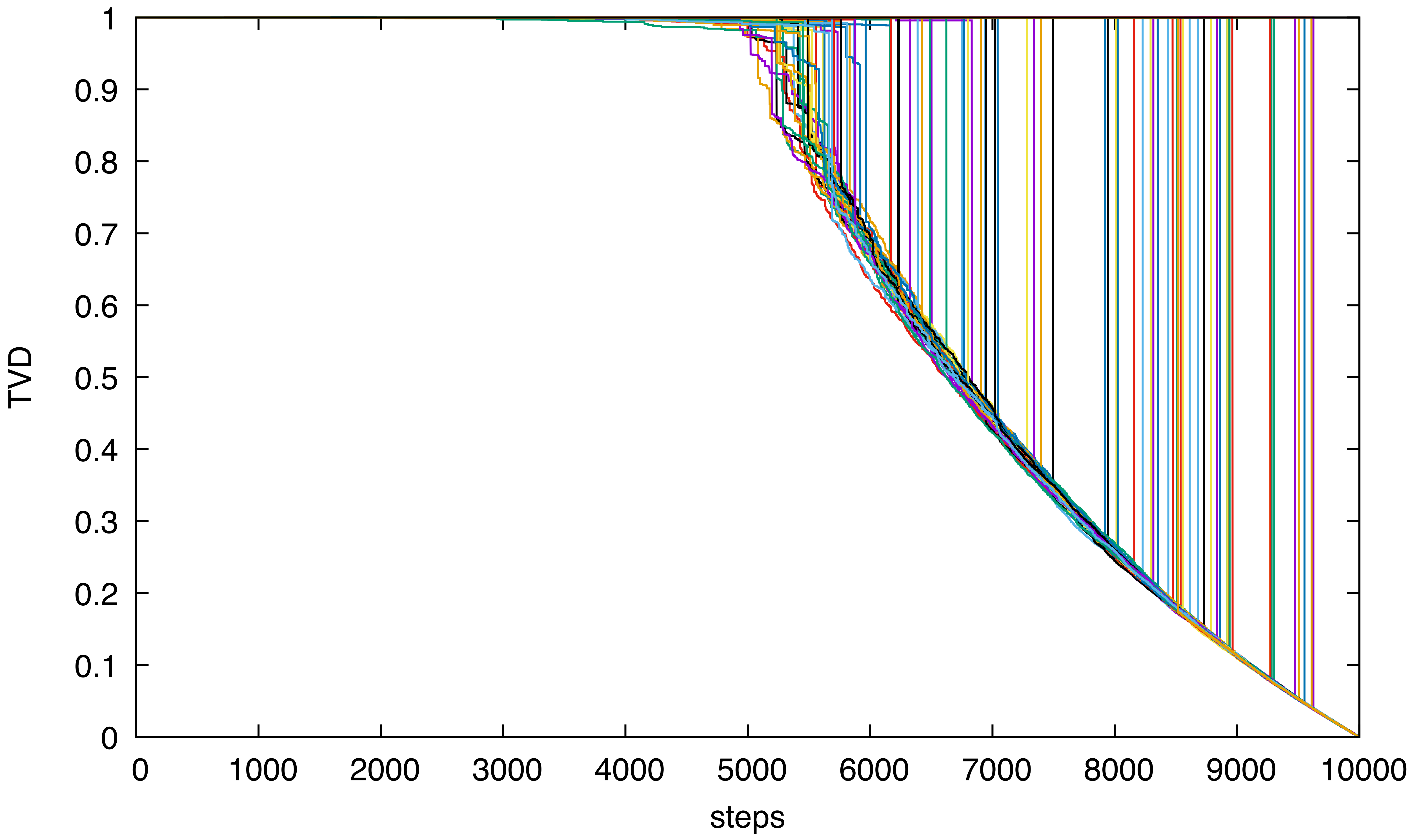
No cut-off



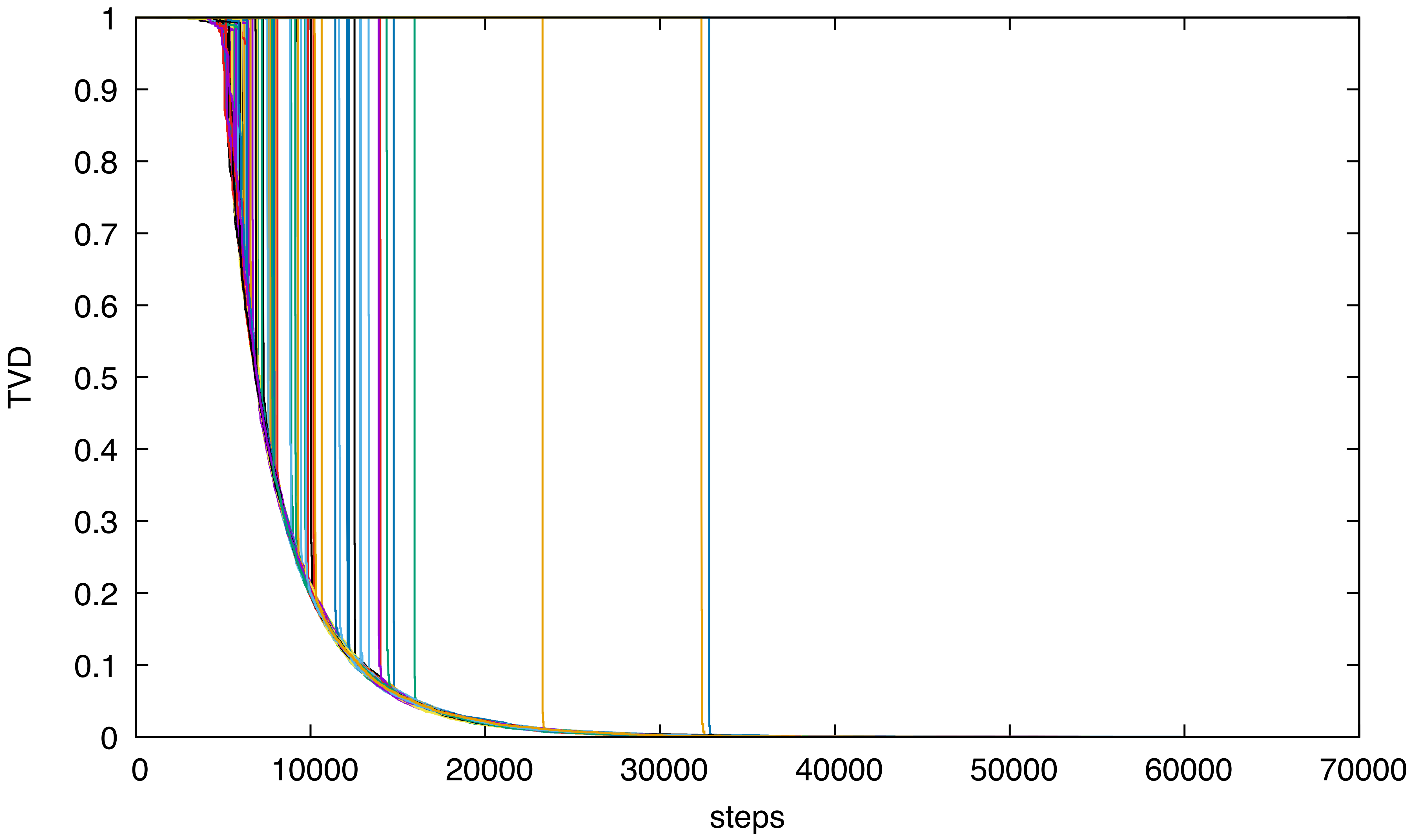
Two-sided cut-off

Results

Simulations



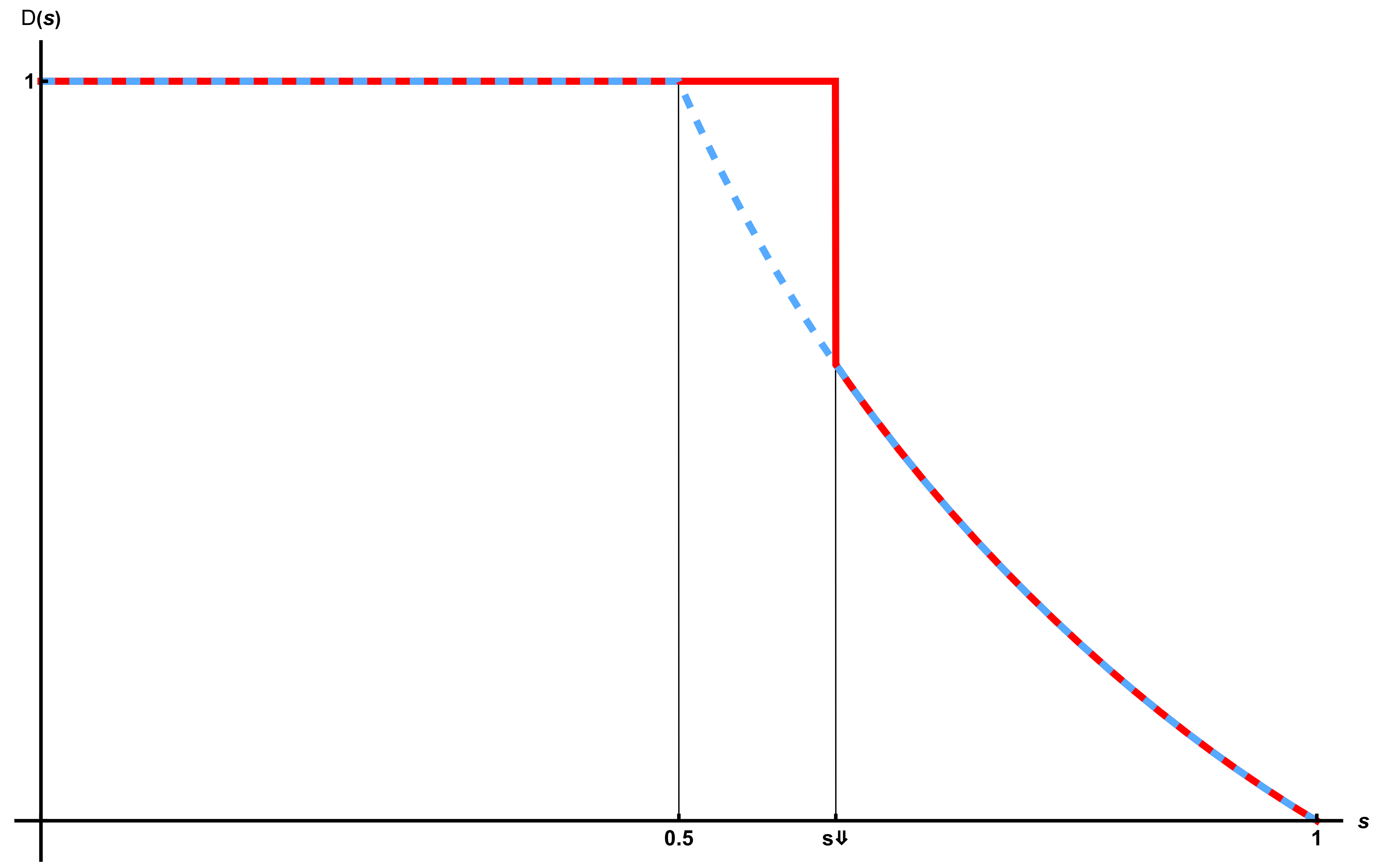
Coagulative dynamics



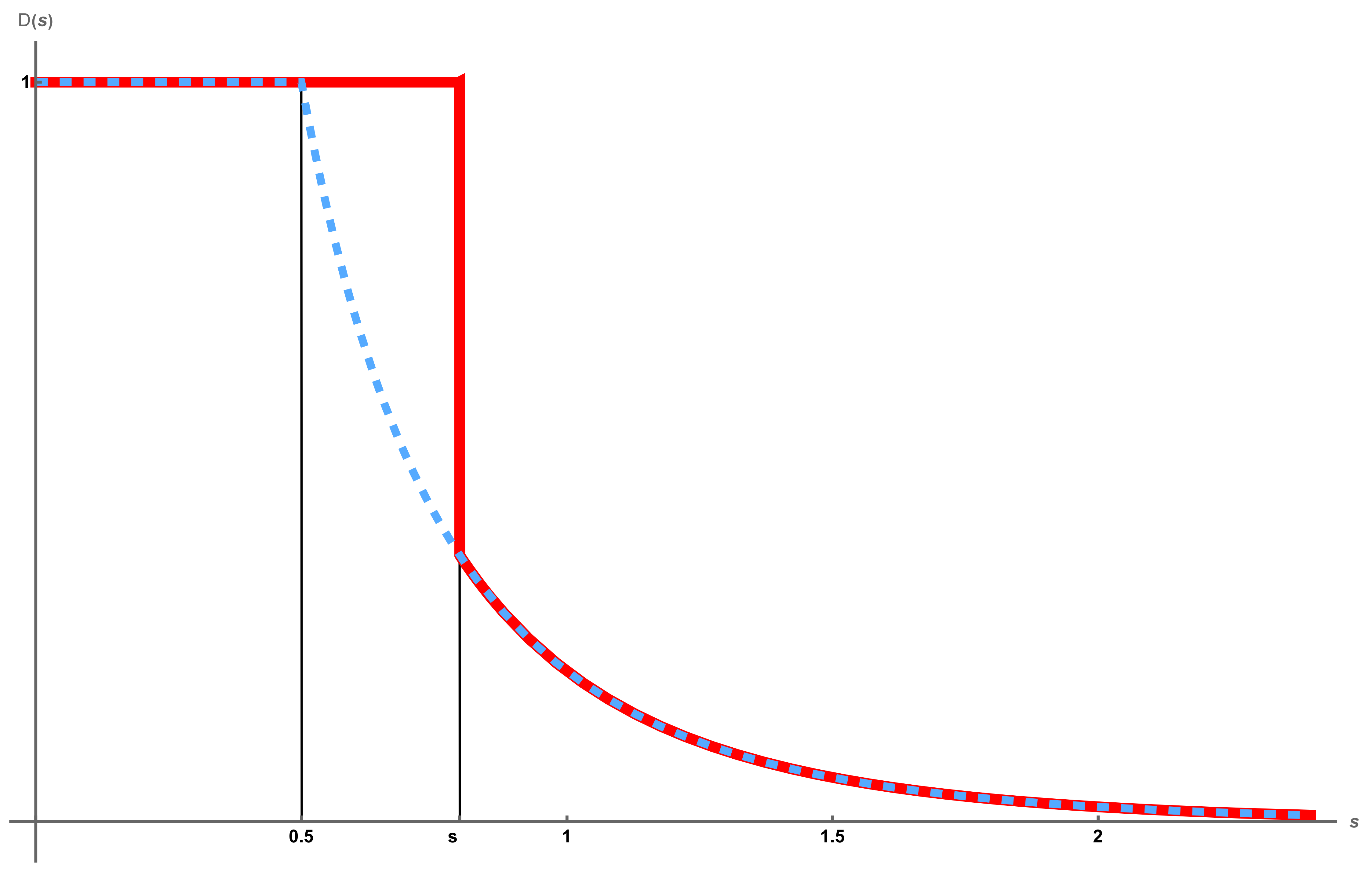
Coagulative-fragmentative dynamics

Results

Plot of a typical realisation



Coagulative dynamics



Coagulative-fragmentative dynamics

Results

Theorem 1.14 [Mixing profile for ISRW on CDP]

(1) *Uniformly in* $v \in [n]$,

$$\frac{T^{\Downarrow}_{n,v}}{n} \xrightarrow{d} s^{\Downarrow}, \tag{1.13}$$

where $s^{\Downarrow}$ *is the* $[0,1]$*-valued random variable with distribution* $\mathbb{P}(s^{\Downarrow} \leq s) = \eta(s)$, $s \in [0,1]$.

(2) *Uniformly in* $v \in [n]$,

$$(\mathcal{D}^v_n(sn))_{s\in[0,1]} \xrightarrow{d} \left(1 - \eta(s)\mathbb{1}_{\{s>s^{\Downarrow}\}}\right)_{s\in[0,1]} \quad \text{in the Skorokhod } M_1\text{-topology.} \tag{1.14}$$

Theorem 1.15 [Mixing profile for ISRW on CFDP]

(1) *Uniformly in* $v \in [n]$,

$$\frac{T^{\Downarrow}_{n,v}}{n} \xrightarrow{d} s^{\Downarrow}, \tag{1.15}$$

where $s^{\Downarrow}$ *is the non-negative random variable with distribution* $\mathbb{P}(s^{\Downarrow} \leq s) = \zeta(s)$, $s \in [0,\infty)$.

(2) *Uniformly in* $v \in [n]$,

$$(\mathcal{D}^v_n(sn))_{s\in[0,\infty)} \xrightarrow{d} \left(1 - \zeta(s)\mathbb{1}_{\{s>s^{\Downarrow}\}}\right)_{s\in[0,\infty)} \quad \text{in the Skorokhod } M_1\text{-topology.} \tag{1.16}$$

3-step path towards the proof

1. Understand the structure and evolution of permutation cycles

2. Identify the distribution of rescaled "drop-down time" $\dfrac{T^{\Downarrow}}{n}$

3. Show "ISRW local mixing" upon drop-down in $o(n)$ steps of the dynamics

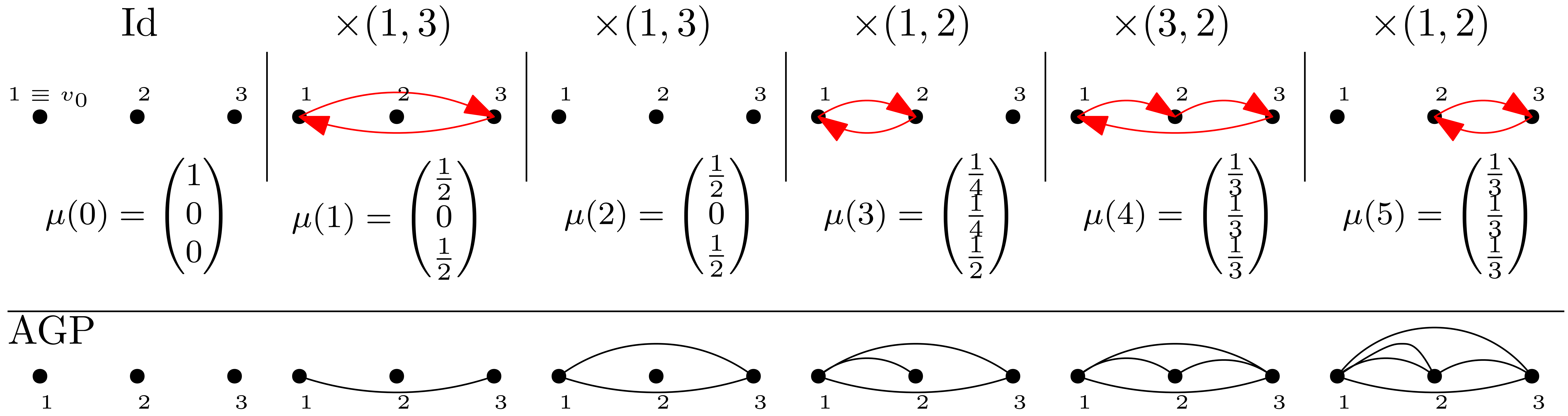
Associated graph process

Definition 2.1 [Graph process associated with Π_n] Let $\Pi_n = (\Pi_n(t))_{t=0}^{t_{\max}}$ with $t_{\max} \in \mathbb{N} \cup \{\infty\}$ be a dynamic permutation starting for the identity permutation. Construct the *associated graph process*, denoted by A_{Π_n}, as follows:

1. At time $t = 0$, start with the empty graph on the vertex set $\mathcal{V} = [n]$.
2. At times $t \in \mathbb{N}$, add the edge $\{a, b\}$, where a, b are such that $\Pi_n(t) = \Pi_n(t-1) \circ (a, b)$.

- Different distributions for different dynamics:
  - Coagulative: Erdős–Rényi with no cycles
  - Coagulative-fragmentative: Erdős–Rényi multigraph with no constraints

Associated graph process



Path towards the proofs

Coagulation-only dynamics

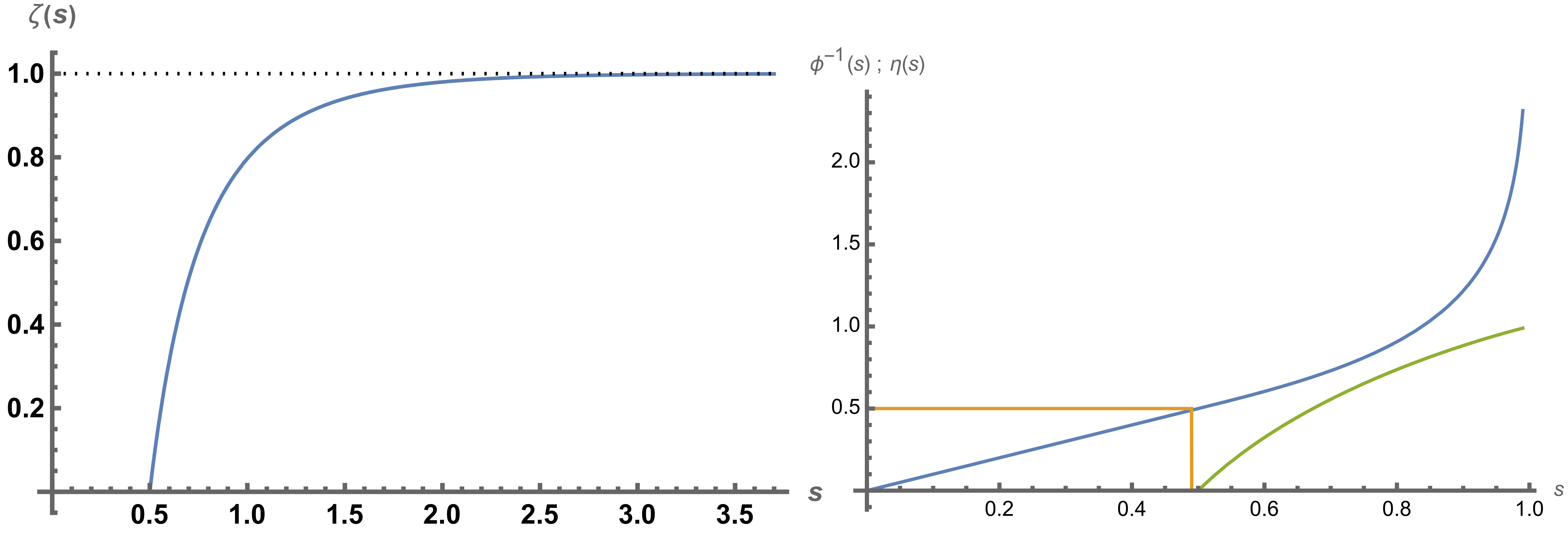
1. **Understand the structure and evolution of permutation cycles**
   - Described by sizes of connected components in the cycle-free Erdős–Rényi model
2. **Identify the distribution of rescaled "drop-down time"** $\dfrac{T^{\Downarrow}}{n}$
   - CDF of $\dfrac{T^{\Downarrow}}{n}$ is given by $\eta(c),\ c \in [0,1]$

     – related to the cycle-free Erdős–Rényi giant component
3. **Show "ISRW local mixing" upon drop-down in $o(n)$ steps of the dynamics**
   - Follows from the definition of the infinite-speed random walk.

Technical problems

Size of the giant component in cycle-free Erdős–Rényi

- Coupling to the unconstrained Erdős–Rényi model $(G(n, M = t))_{t=0}^{\infty}$
- Sizes of connected components of $F_n(t)$ correspond to the sizes of connected components of $G_n(\tau)$ with some different time $\tau(t)$.



Path towards the proofs

Coagulative-fragmentative dynamics

1. **Understand the structure and evolution of permutation cycles**

    - Schramm (2005): $\mathrm{PD}(1)$ substructure on the giant

2. **Identify the distribution of rescaled "drop-down time" $\dfrac{T^{\Downarrow}}{n}$**

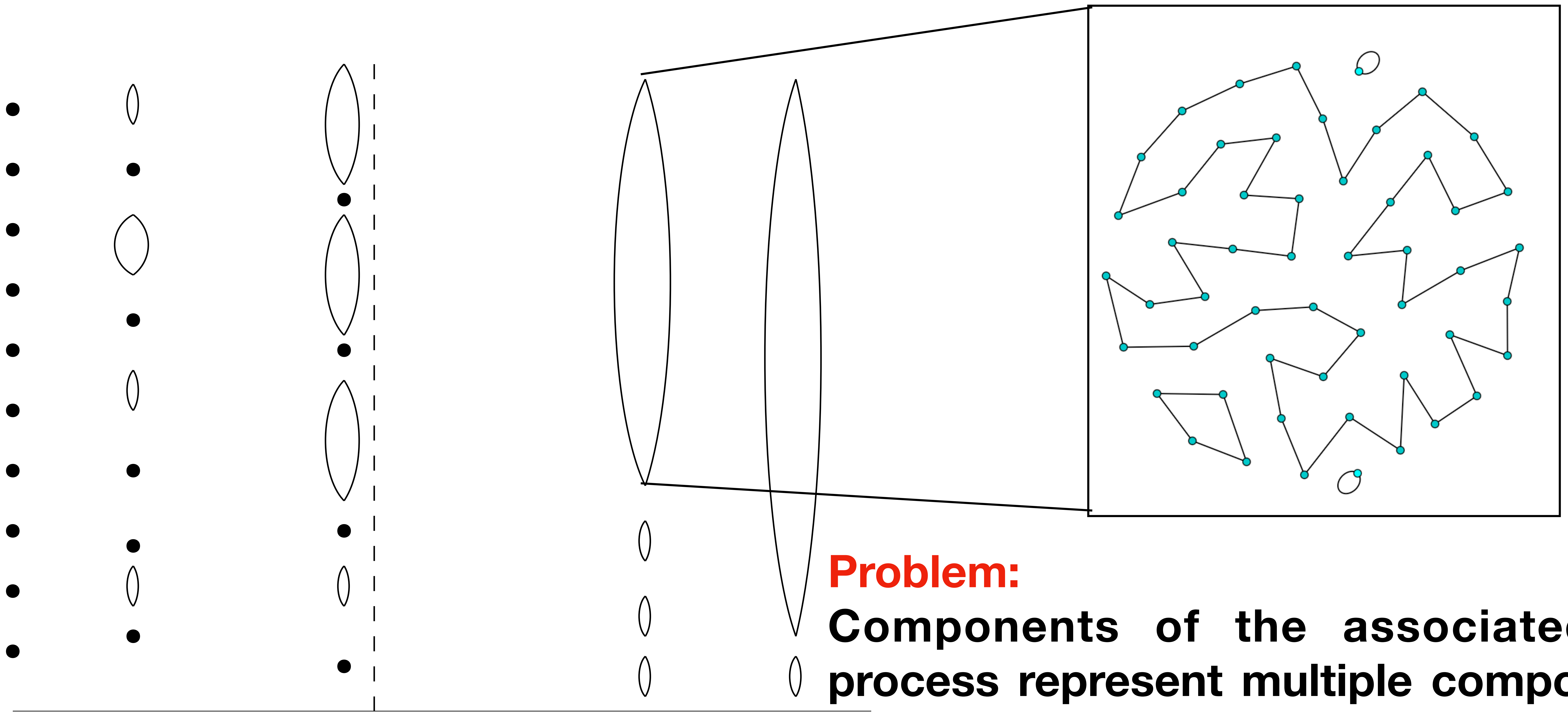
    - CDF of $T^{\Downarrow}/n$ is given by $\zeta(c),\ c \in (0,\infty)$ – related to the Erdős–Rényi giant component.
    - Furthermore, whp the support of the ISRW does not experience fragmentation before $T^{\Downarrow}$

3. **Show "ISRW local mixing" upon drop-down in $o(n)$ steps of the dynamics**

    - Recurrence of large-enough permutation cycles on the AGP-giant implies mixing

Technical problems

Mixing in sublinear-time upon drop-down for coag.-frag. dynamics



Problem:

Components of the associated graph process represent multiple components of the underlying permutation!

Technical problems

Mixing in sublinear-time upon drop-down for coag.-frag. dynamics

- By definition of the infinite-speed random walk, all the mass on any permutation cycle gets spread out uniformly over that cycle.
- Therefore ε-mixing on the giant ($\equiv \|X(t) - \text{Uniform}([\,|\mathscr{C}^{\max}_{\text{AGP}}(t)|\,])\|_{\text{TV}} \leq \varepsilon$) can be achieved by spreading over cycle C s. t. $|C(t)| = (1-\varepsilon)|\mathscr{C}^{\max}_{\text{AGP}}(t)|$
- **How to show recurrence of these large cycles?**
  - Show that the evolution of the cycle structure can be well-approximated by a Markov chain with some “nice” properties.

Technical problems

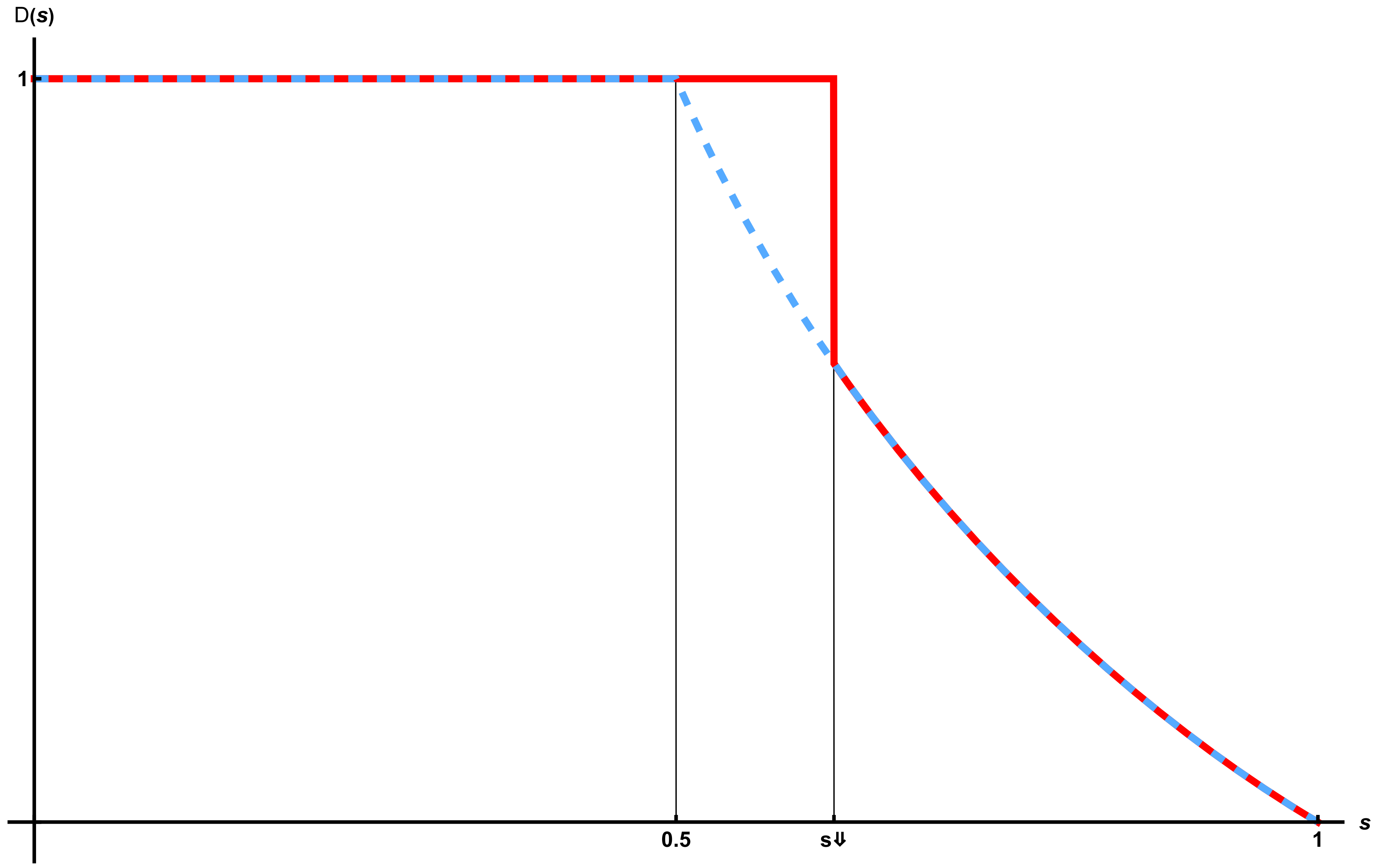
Recurrence of large-enough cycles

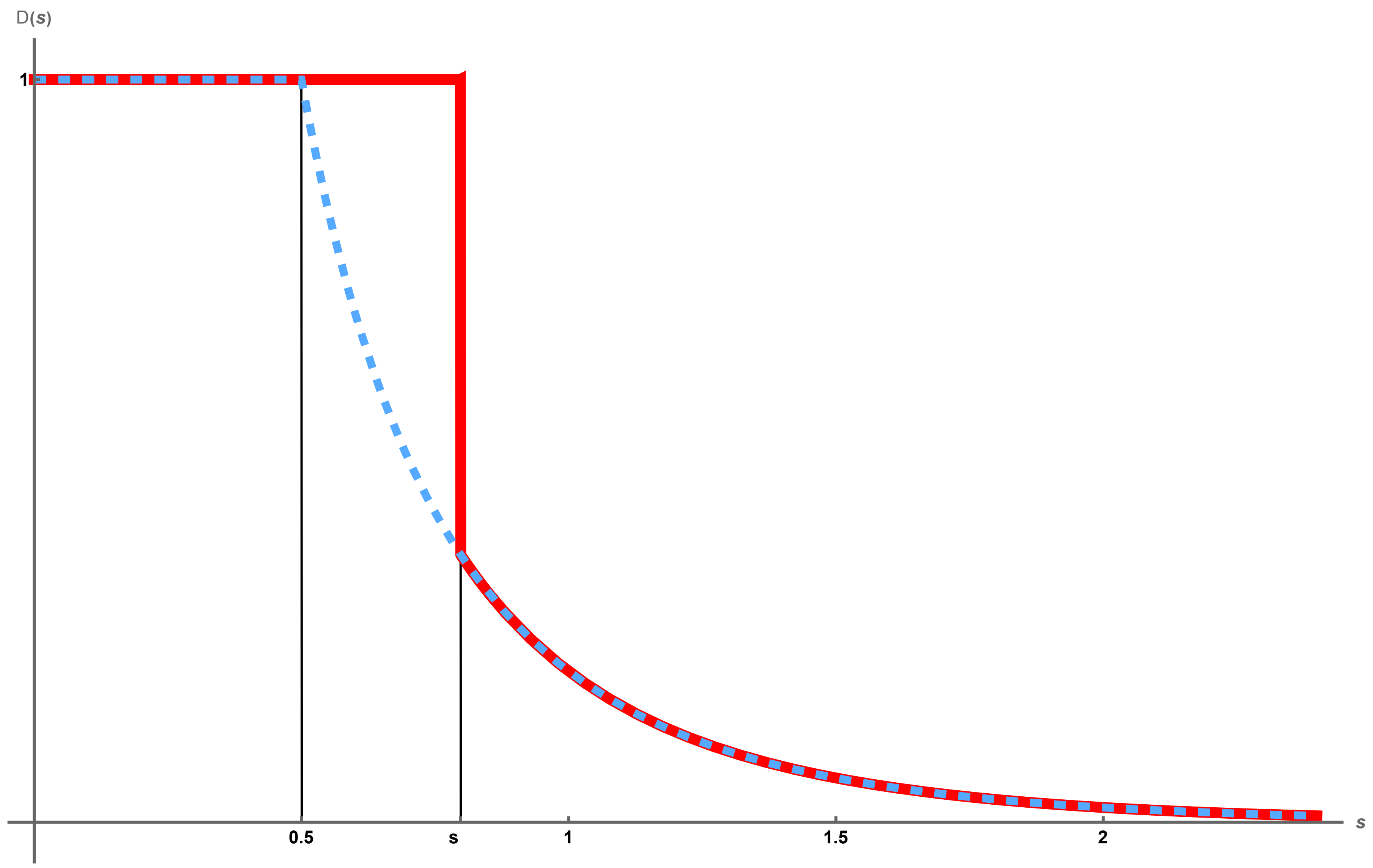
- Schramm's coupling:
  - Coupling between the cycle structure of $\Pi_n(t),\ t > cn,\ c > 1/2$ and a sample from PD(1).
  - Under this coupling the permutation cycle structure and a PD(1) sample get close in sup-norm. Under some fairly mild assumptions, for any $\varepsilon > 0$ and $q \sim$ Uniform$(2\mathbb{Z} \cap [0, \lfloor \varepsilon^{-1/2} \rfloor])$, it holds that:

$$\mathbf{P}[\|Y^q - Z^q\|_\infty > \rho] \leq O(1)\rho^{-1} |\log \epsilon|^{-1}. \tag{4.1}$$

Summary

- ISRW on dynamic permutations has a TVD-mixing profile with a discontinuity at a random time.





- Possibly an interesting dynamic geometry for other stochastic models, such as the voter model.

References

- **Cycle structure of dynamic random permutations**
  - N. Berestycki and R. Durrett. A phase transition in the random transposition random walk. Probab. Theory Relat. fields, 136:203–233, 2006.
  - O. Schramm. Compositions of random transpositions. Israel J. Math., 147:221–243, 2005.
- **Accesible explanation of Schramm's coupling**
  - J. E. Björnberg, M. Kotowski, B. Lees, and P. Miłoś. The interchange process with reversals on the complete graph. Electron. J. Prob., 24:1–43, 2019.

Thank you for your attention.