Mixing of fast random walks on dynamic random permutations

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Model

Dynamic permutation

- Take $n \in \mathbb{N}$ and define the sequence $\left(\Pi_n(t)\right)_{t=0}^{\infty}$ such that:
  - $\Pi_n(0) = \mathtt{Id} \in S_n$
  - $\forall t \geq 1 : \Pi_n(t) = \Pi_n(t-1) \circ (a, b)$,
    where (a, b) is transposition chosen according to a given rule
- Dynamic rules under consideration:
  - Transpositions of elements on different cycles picked u.a.r. (coagulation-only)
  - Transpositions chosen u.a.r. (coagulation-fragmentation)

Model

Infinite-speed random walk

Definition 1.6 [Infinite-speed random walk on Π_n] Fix Π_n and an element $v_0 \in [n]$. Recall that $\gamma_v(\Pi_n(t))$ is the cycle of $\Pi_n(t)$ that contains v. The infinite-speed random walk (ISRW) starting from v_0 is the random process $X_n^{v_0} = (X_n^{v_0}(t))_{t \in \mathbb{N}_0}$ on $[n]$ with initial distribution given by

$$\mu^{X_n^{v_0}}(0) = \left(\mu_w^{X_n^{v_0}}(0)\right)_{w \in [n]}, \tag{1.4}$$

where

$$\mu_w^{X_n^{v_0}}(0) = \begin{cases} \frac{1}{|\gamma_w(\Pi_n(0))|}, & w \in \gamma_{v_0}(\Pi_n(0)), \\ 0, & w \notin \gamma_{v_0}(\Pi_n(0)), \end{cases} \tag{1.5}$$

and with distribution at time $t \in \mathbb{N}$ given by

$$\mu^{X_n^{v_0}}(t) = \left(\mu_w^{X_n^{v_0}}(t)\right)_{w \in [n]}, \tag{1.6}$$

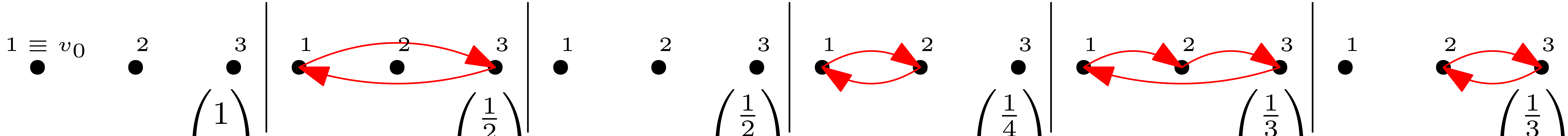
where

$$\mu_w^{X_n^{v_0}}(t) = \frac{1}{|\gamma_w(\Pi_n(t))|} \sum_{u \in \gamma_w(\Pi_n(t))} \mu_u^{X_n^{v_0}}(t-1). \tag{1.7}$$

Model

Infinite-speed random walk on a dynamic permutation

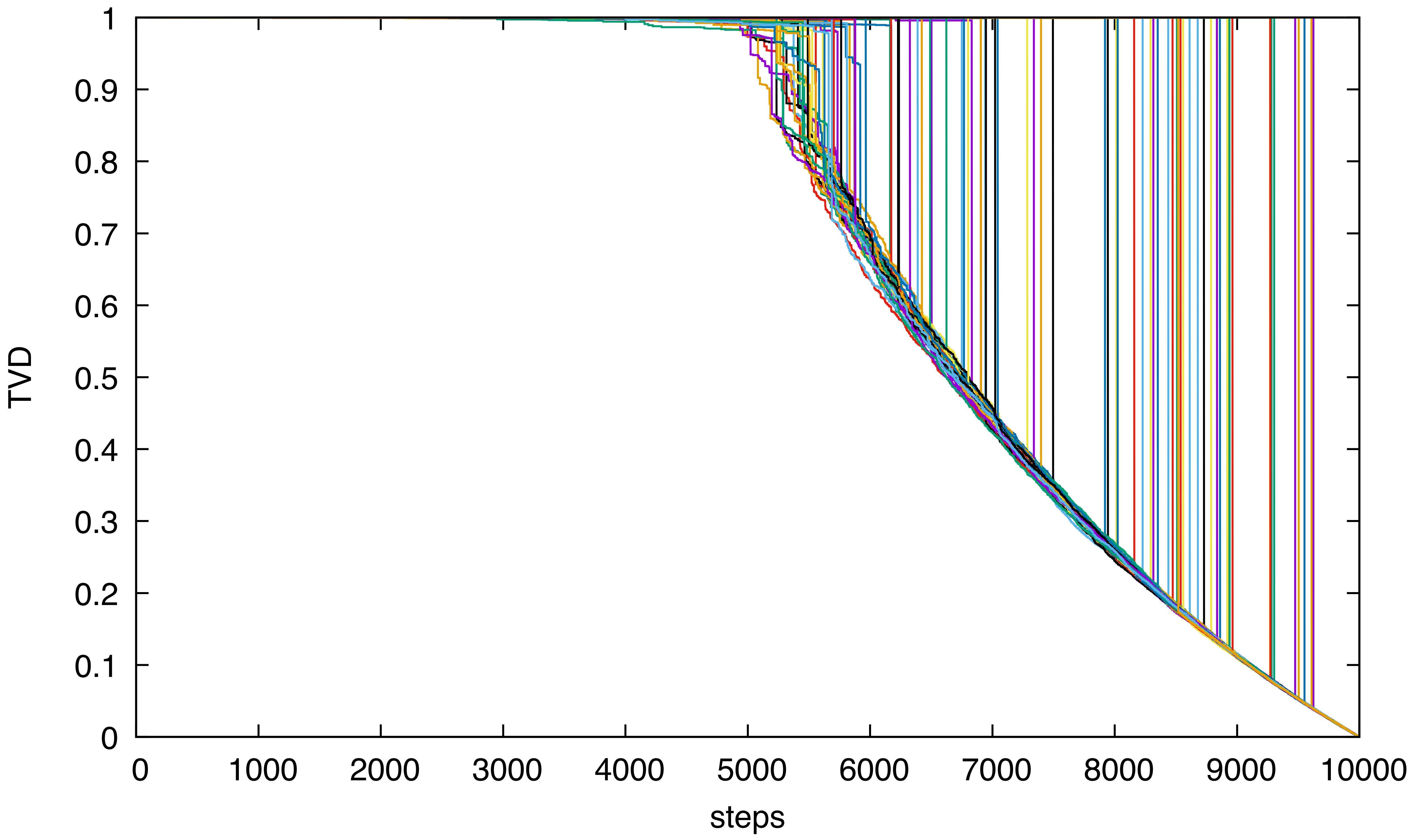
$$\mathrm{Id} \qquad \circ(1,3) \qquad \circ(1,3) \qquad \circ(1,2) \qquad \circ(3,2) \qquad \circ(1,2)$$



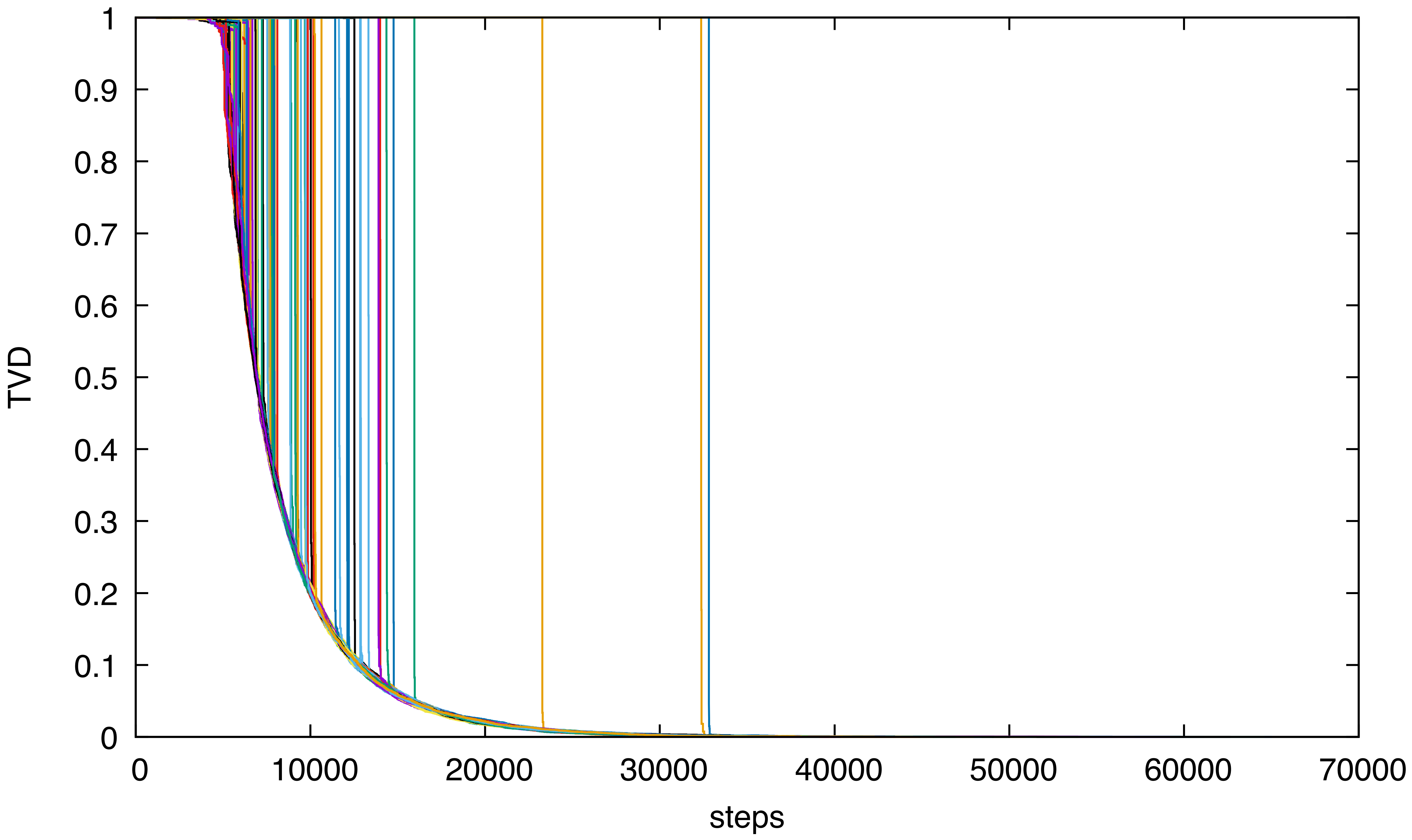
$$\mu^{X_n^{v_0}}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mu^{X_n^{v_0}}(1) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \mu^{X_n^{v_0}}(2) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \mu^{X_n^{v_0}}(3) = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix} \quad \mu^{X_n^{v_0}}(4) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} \quad \mu^{X_n^{v_0}}(5) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

Results

Simulations



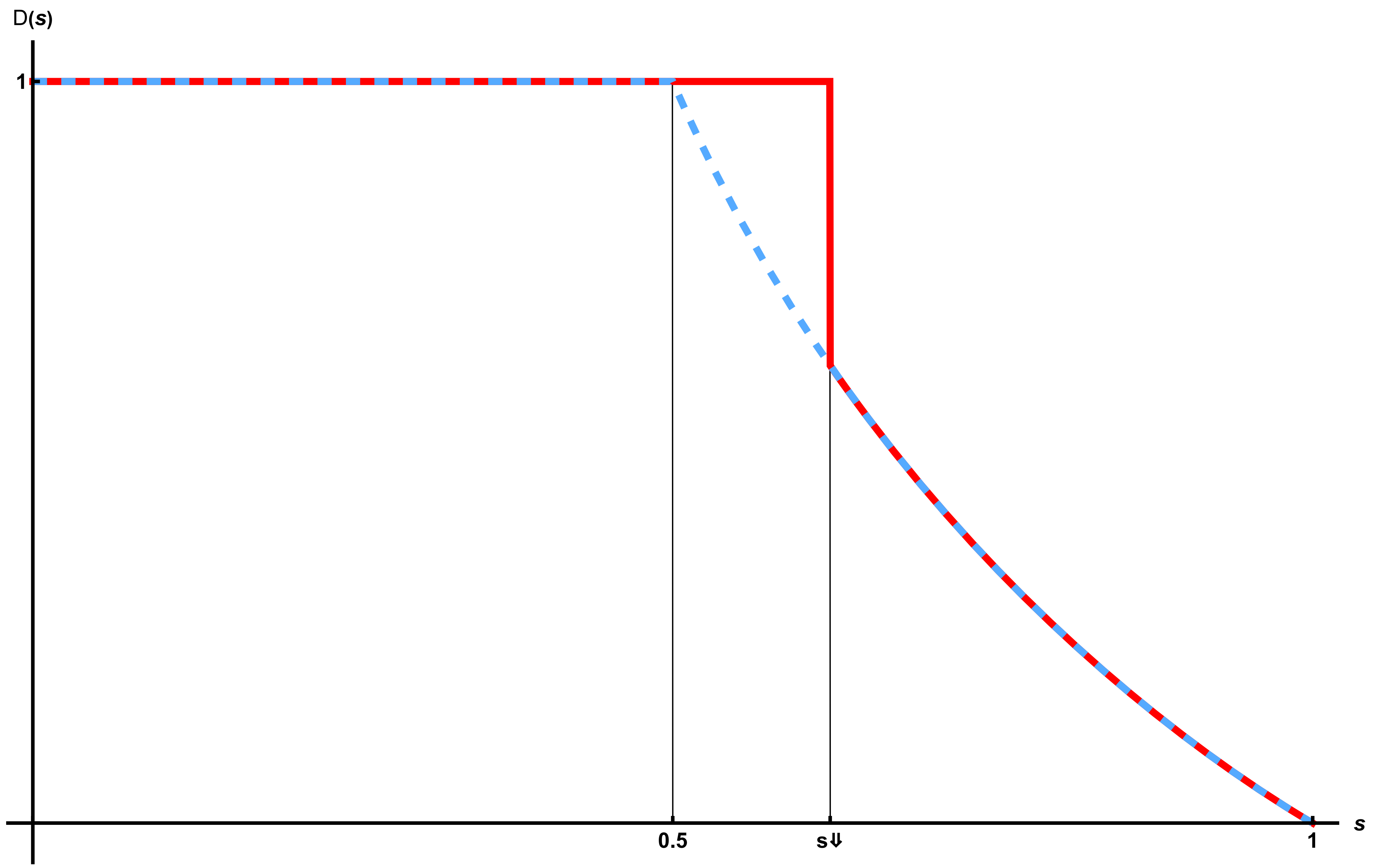
Coagulative dynamics



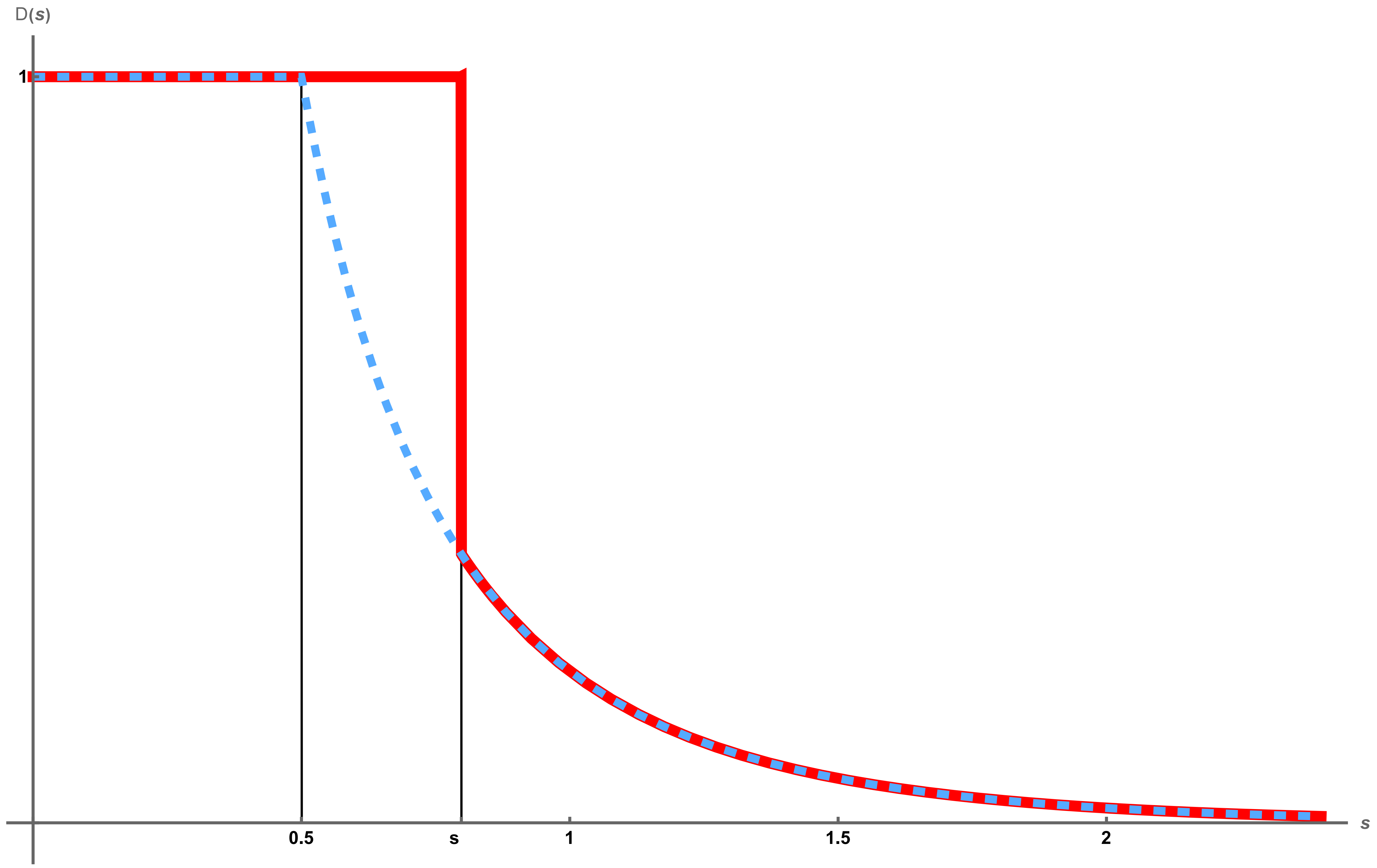
Coagulative-fragmentative dynamics

Results

Plot of a typical realisation



Coagulative dynamics



Coagulative-fragmentative dynamics

Results

Theorem 1.14 [Mixing profile for ISRW on CDP]

(1) *Uniformly in* $v \in [n]$,

$$\frac{T^{\Downarrow}_{n,v}}{n} \xrightarrow{d} s^{\Downarrow}, \tag{1.13}$$

where $s^{\Downarrow}$ *is the* $[0,1]$*-valued random variable with distribution* $\mathbb{P}(s^{\Downarrow} \leq s) = \eta(s)$, $s \in [0,1]$.

(2) *Uniformly in* $v \in [n]$,

$$(\mathcal{D}^v_n(sn))_{s\in[0,1]} \xrightarrow{d} \left(1 - \eta(s)\mathbb{1}_{\{s>s^{\Downarrow}\}}\right)_{s\in[0,1]} \quad \textit{in the Skorokhod } M_1\textit{-topology.} \tag{1.14}$$

Theorem 1.15 [Mixing profile for ISRW on CFDP]

(1) *Uniformly in* $v \in [n]$,

$$\frac{T^{\Downarrow}_{n,v}}{n} \xrightarrow{d} s^{\Downarrow}, \tag{1.15}$$

where $s^{\Downarrow}$ *is the non-negative random variable with distribution* $\mathbb{P}(s^{\Downarrow} \leq s) = \zeta(s)$, $s \in [0,\infty)$.

(2) *Uniformly in* $v \in [n]$,

$$(\mathcal{D}^v_n(sn))_{s\in[0,\infty)} \xrightarrow{d} \left(1 - \zeta(s)\mathbb{1}_{\{s>s^{\Downarrow}\}}\right)_{s\in[0,\infty)} \quad \textit{in the Skorokhod } M_1\textit{-topology.} \tag{1.16}$$

Associated graph process

Definition 2.1 [Graph process associated with Π_n] Let $\Pi_n = (\Pi_n(t))_{t=0}^{t_{\max}}$ with $t_{\max} \in \mathbb{N} \cup \{\infty\}$ be a dynamic permutation starting for the identity permutation. Construct the *associated graph process*, denoted by A_{Π_n}, as follows:

1. At time $t = 0$, start with the empty graph on the vertex set $\mathcal{V} = [n]$.
2. At times $t \in \mathbb{N}$, add the edge $\{a, b\}$, where a, b are such that $\Pi_n(t) = \Pi_n(t-1) \circ (a, b)$.

- Different distributions for different dynamics:
  - For coagulative-only: Erdős–Rényi with no cycles
  - For coagulative-fragmentative: Erdős–Rényi multigraph with no constraints

Path towards the proofs

Coagulation-only dynamics

- **Understand evolution of permutation cycles**
  - Described by sizes of connected components in the cycle-free Erdős–Rényi model
- **Identify the distribution of rescaled “drop-down time”** $T^{\Downarrow}/n$
  - CDF of $\dfrac{T^{\Downarrow}}{n}$ is given by $\eta(c),\ c \in [0,1]$

    – related to the cycle-free Erdős–Rényi giant component
- **Show mixing properties**
  - Follows from the definition of the infinite-speed random walk

Path towards the proofs

Coagulative-fragmentative dynamics

- **Understand evolution of permutation cycles**
  - Schramm (2005):
    coupling to Erdős–Rényi multigraph, $\mathrm{PD}(1)$ substructure on the giant
- **Identify the distribution of rescaled “drop-down time” $T^{\Downarrow}/n$**
  - CDF of $T^{\Downarrow}/n$ is given by $\zeta(c),\ c \in (0,\infty)$ – related to the Erdős–Rényi giant component.
  - Furthermore, whp the support of the ISRW does not experience fragmentation before $T^{\Downarrow}$
- **Show mixing properties**
    - Recurrence of large-enough permutation cycles on the AGP-giant imply mixing

Technical problems

Size of the giant component in cycle-free Erdős–Rényi

- Coupling to the unconstrained Erdős–Rényi model:

Definition 2.3 [Coupling between cycle-free and standard Erdős-Rényi graph process] Let $G_n = (G_n(t))_{t\in\mathbb{N}_0}$ be the Erdős-Rényi graph process on $[n]$ defined in Definition 1.9, and denote the edge set of $G_n(t)$ by $\mathcal{E}_{G_n(t)}$. Based on G_n, construct a graph-valued process $F_n = (F_n(t))_{t\in\mathbb{N}_0}$ as follows:

1. $F_n(0)$ is the empty graph with vertex set $[n]$.
2. At times $t \in \mathbb{N}$, define $e^c(t) = \mathcal{E}_{G_n(t)} \setminus \mathcal{E}_{G_n(t-1)}$, which is the edge added at time t to $G_n(t)$.
   (a) Construct the candidate graph at time t, defined as $F_n^c(t) = (\mathcal{V}, \mathcal{E}_{F_n(t-1)} \cup \{e^c(t)\})$.
   (b) If $F_n^c(t)$ is a forest, then set $F_n(t) = F_n^c(t)$.
   (c) Otherwise, set $F_n(t) = F_n(t-1)$.

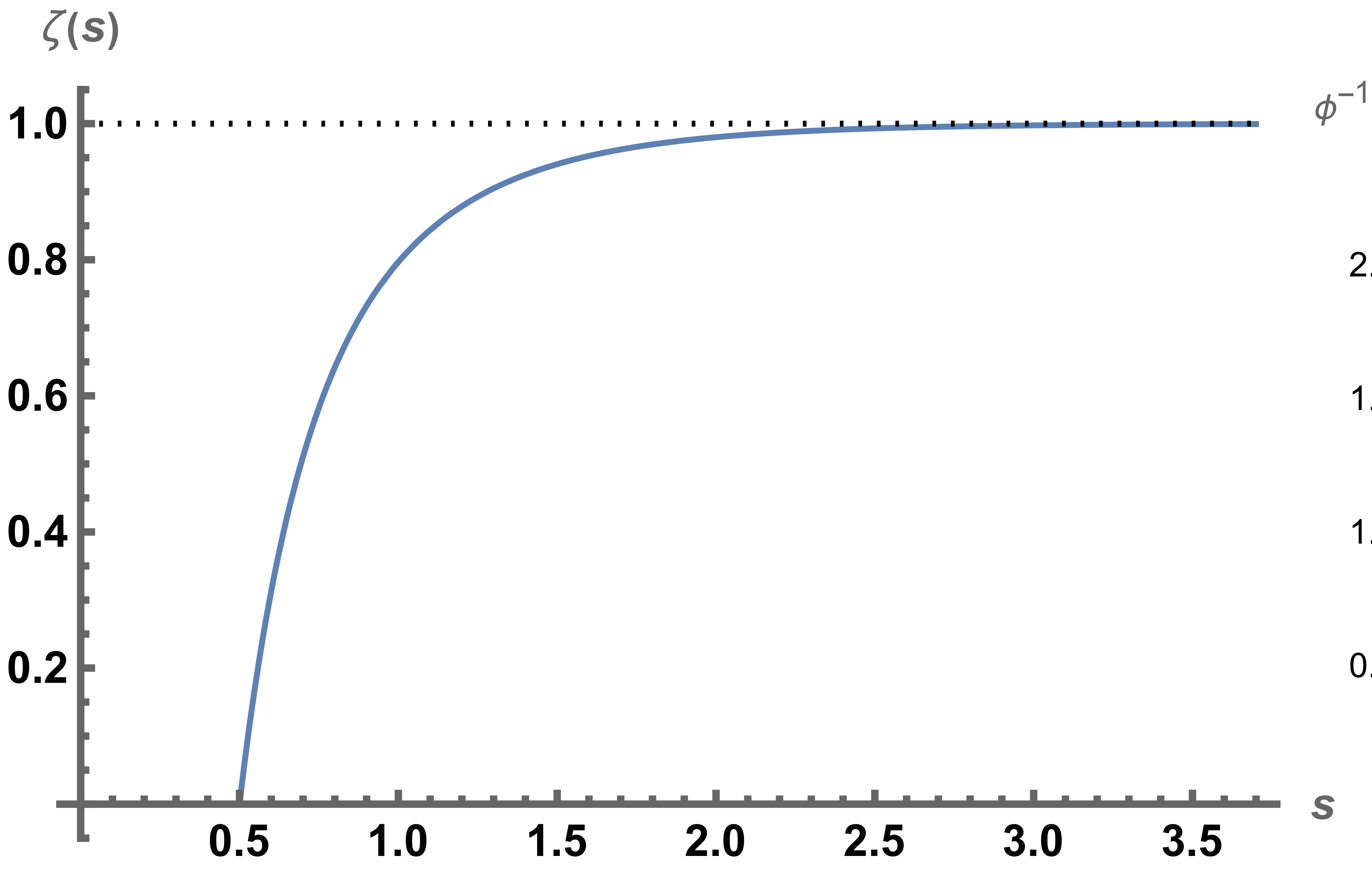
Define the *effective time* $\tau_n(t)$ of the coupled process $(F_n(t))_{t\in\mathbb{N}_0}$ by setting $\tau_n(0) = 0$ and, recursively for $t \in \mathbb{N}$,

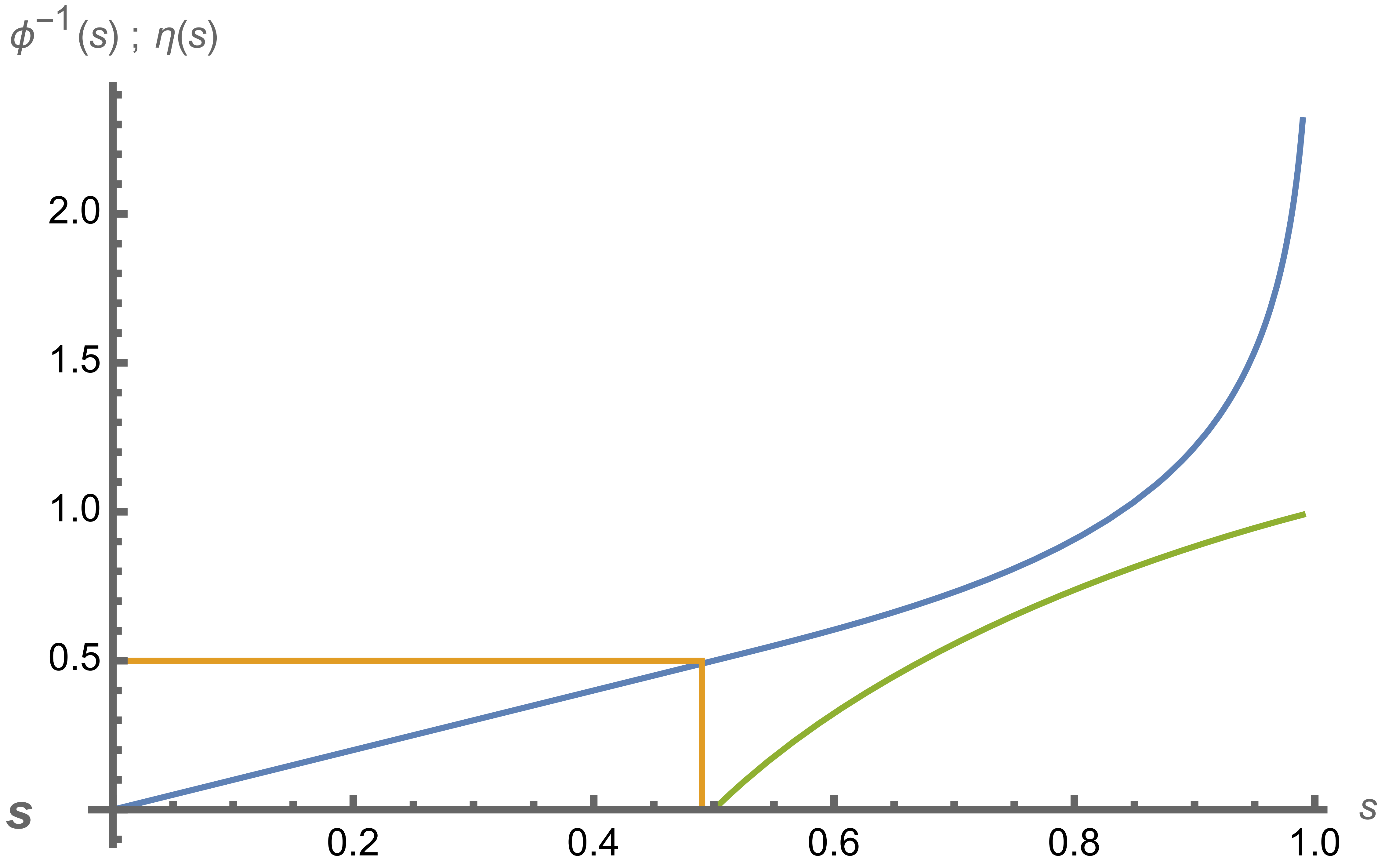
$$\tau_n(t) = \begin{cases} \tau_n(t-1) + 1, & \text{if } F_n(t) \neq F_n(t-1), \text{ i.e., the proposed edge has been accepted,} \\ \tau_n(t-1), & \text{if } F_n(t) = F_n(t-1), \text{ i.e., the proposed edge has been rejected.} \end{cases} \quad (2.1)$$

- Sizes of connected components of $F_n(t)$ correspond to the sizes of connected components of $G_n(\tau)$ with some different time $\tau(t)$.

Technical problems

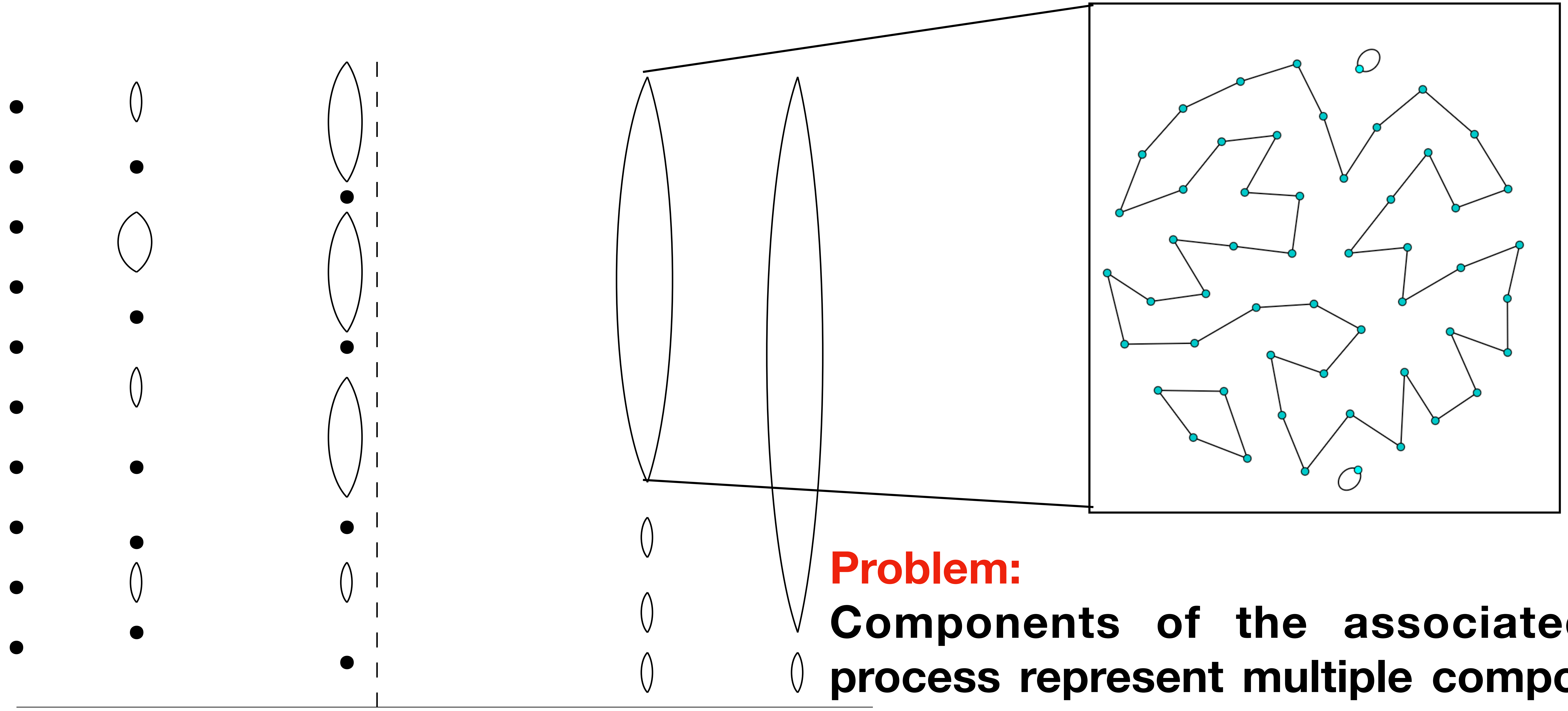
Size of the giant component in cycle-free Erdős–Rényi





Technical problems

Mixing in sublinear-time upon drop-down for coag.-frag. dynamics



Problem:

Components of the associated graph process represent multiple components of the underlying permutation!

Technical problems

Mixing in sublinear-time upon drop-down for coag.-frag. dynamics

- By definition of the infinite-speed random walk, all the mass on any permutation cycle gets spread out uniformly over that cycle.
- Therefore ε-mixing on the giant ($\equiv \|X(t) - \text{Uniform}([\,|\mathscr{C}^{\max}_{\text{AGP}}(t)|\,])\|_{\text{TV}} \leq \varepsilon$) can be achieved by spreading over cycle C s. t. $|C(t)| = (1-\varepsilon)|\mathscr{C}^{\max}_{\text{AGP}}(t)|$
- **How to show recurrence of these large cycles?**

Technical problems

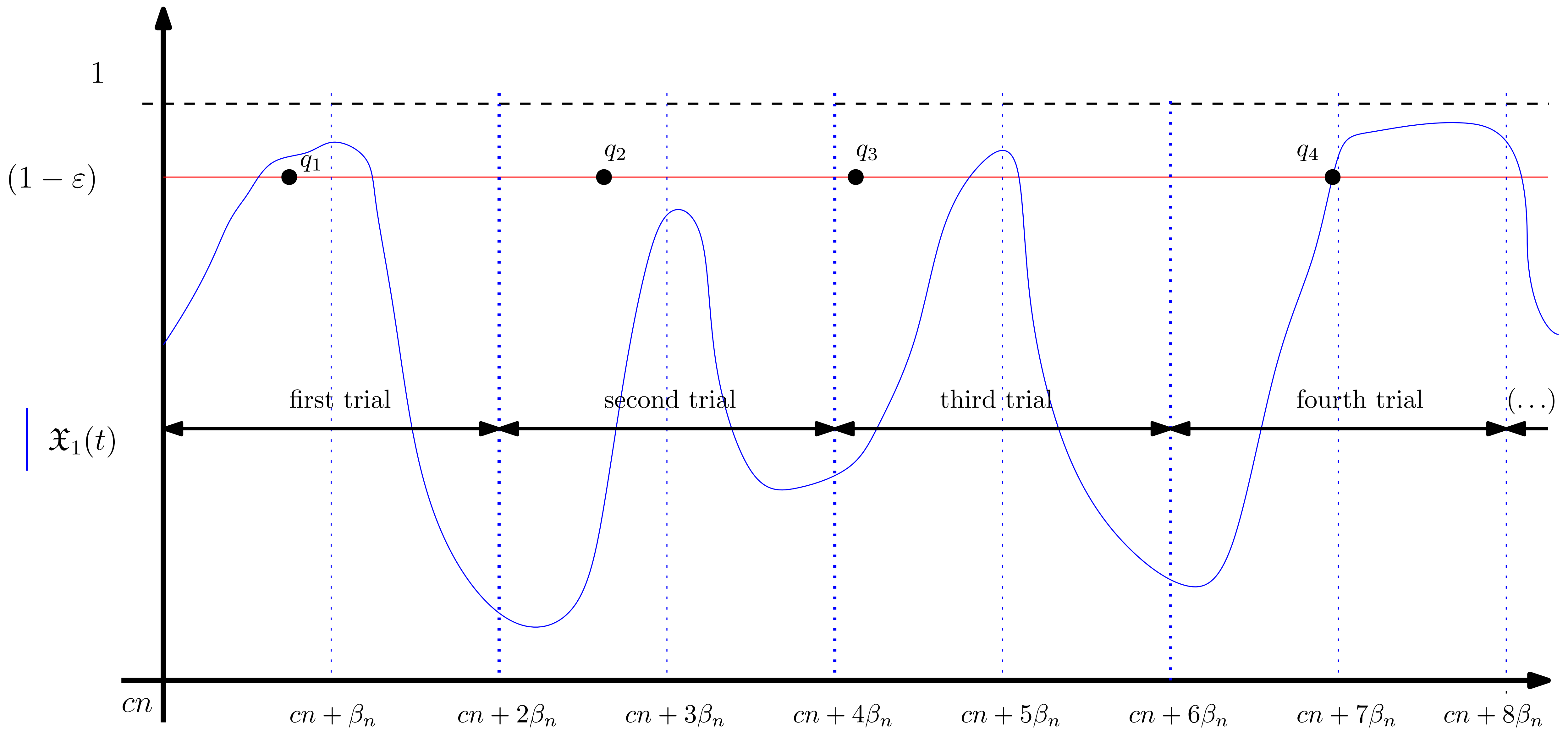
Recurrence of large-enough cycles

- Schramm's coupling:
  - Coupling between the cycle structure of $\Pi_n(t),\ t > cn,\ c > 1/2$ and a sample from PD(1).
  - Under this coupling the permutation cycle structure and a PD(1) sample get close in sup-norm. Under some fairly mild assumptions, $\varepsilon > 0$ and $q \sim$ Uniform$(2\mathbb{Z} \cap [0, \lfloor \varepsilon^{-1/2} \rfloor])$. Then:

$$\mathbf{P}[\|Y^q - Z^q\|_\infty > \rho] \le O(1)\rho^{-1}|\log \epsilon|^{-1}. \tag{4.1}$$

Technical problems

Trials using Schramm's coupling



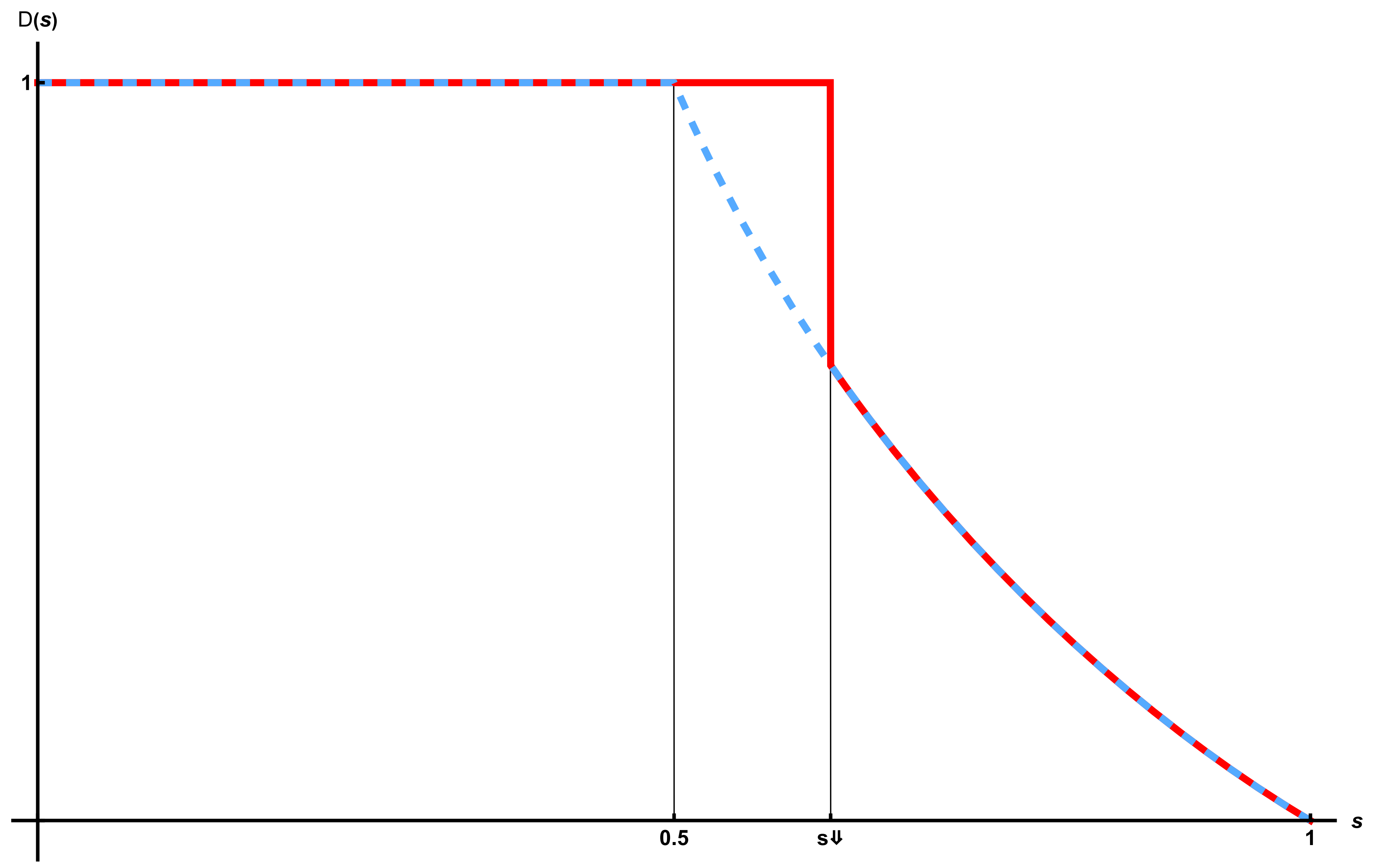
- In one trial:

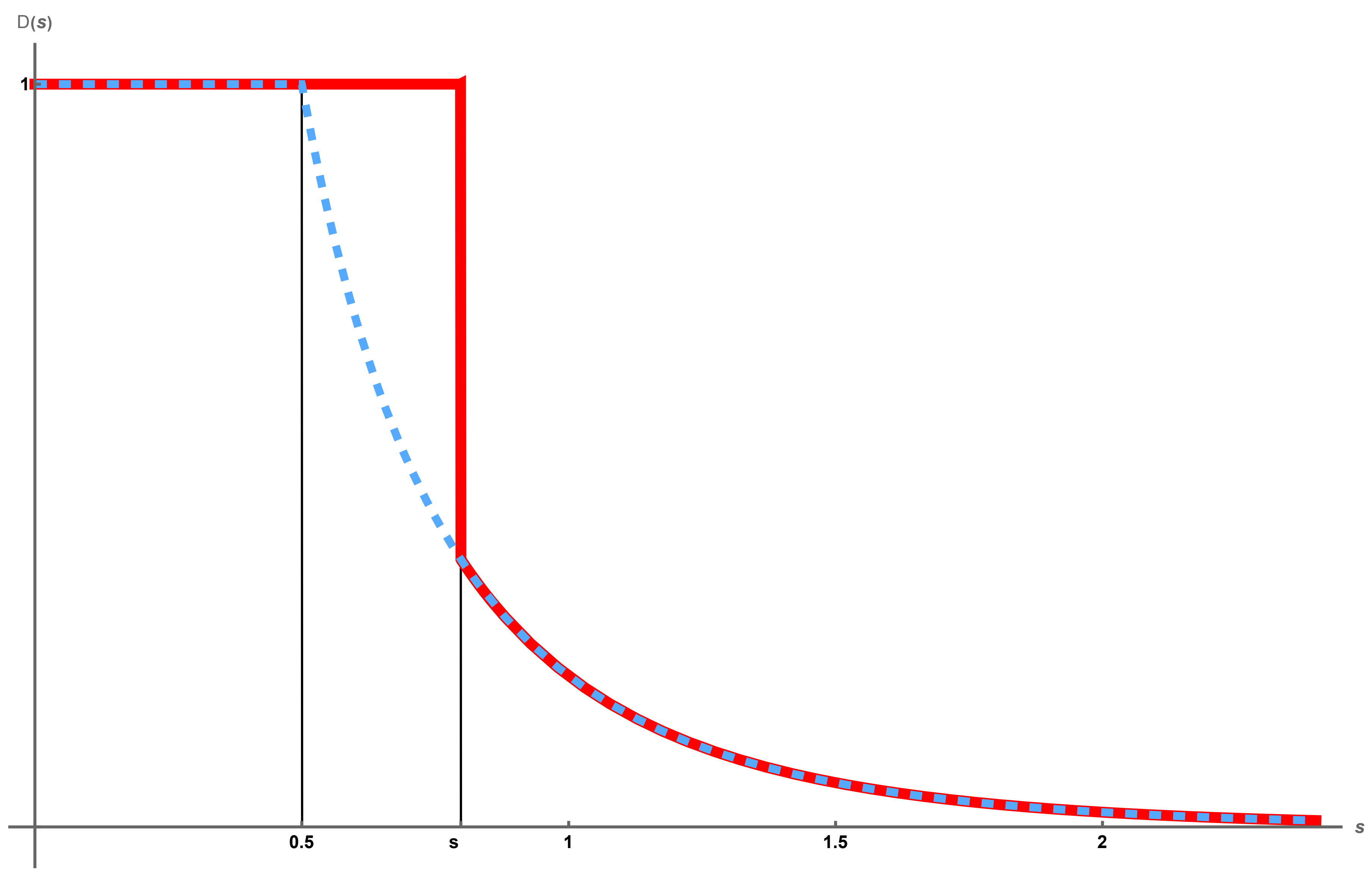
1. Sample an independent $Z_i(t) \sim \mathrm{PD}(1)$
2. Use Schramm's coupling to couple it to the underlying dynamic permutation
3. Wait for the possible large component to break down
4. Repeat

- It can be shown that there is a strictly positive probability that a large-enough appeared during the trial and subsequently broke down.

Summary

- ISRW on dynamic permutations has a TVD-mixing profile with a discontinuity at a random time.





- Possibly an interesting dynamic geometry for other stochastic models, such as the voter model.

References

- **Cycle structure of dynamic random permutations**

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  - O. Schramm. Compositions of random transpositions. Israel J. Math., 147:221–243, 2005.

- **Accesible explanation of Schramm’s coupling**

  - J. E. Björnberg, M. Kotowski, B. Lees, and P. Miłoś. The interchange process with reversals on the complete graph. Electron. J. Prob., 24:1–43, 2019.

Thank you for your attention.